EC8391 - CONTROL SYSTEMS ENGINEERING

UNIT 1

SYSTEM COMPONENTS AND THEIR REPRESENTATION

Control System: Terminology and Basic Structure - Feed forward and Feedback control theory - Electrical and Mechanical Transfer function Models - Block diagram Models - Signal flow graphs models - DC and AC servo systems - Synchronous - Multivariable control system

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INTRODUCTION

Control Systems deal with control of Engineering systems that are governed by the laws of physics and are therefore called Physical systems.

The word control means to regulate, to direct or to command. The word system means a combination of devices and components connected together, to perform a certain function. This system may be physical, biological, economic etc.

Control System is defined as combination of devices and components connected or related so as to command, direct or regulate itself or another system.

It is used in many applications. eg:- control of temperature, liquid level, position, velocity, flow pressure, etc.,

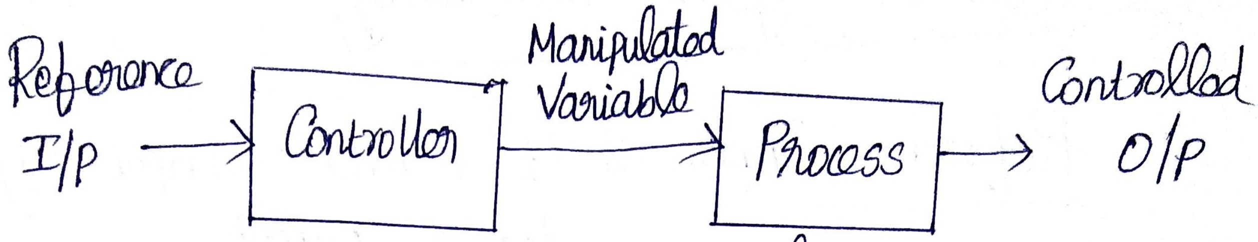
Classification of control system

Open Loop control system [Feed forward]

Closed loop control system [Feedback]

Open Loop Control System [Feed forward]

A System in which the control action is totally independent of the output of the system is called as open loop system.



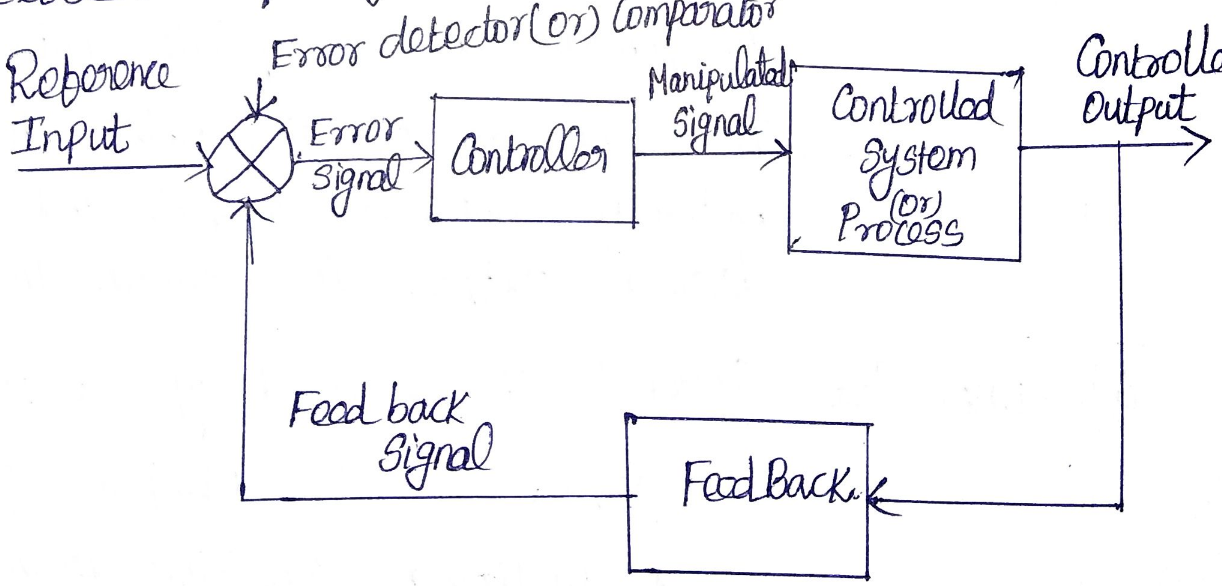
Block diagram of open loop system

Example

Electric hand drier - Hot air (output) comes out as long as you keep your hand under the machine, irrespective of how much your hand is dried.

Closed loop Control System (feedback)

A system in which the control action is somehow dependent on the output is called as closed loop system.



Block diagram of closed loop control system

Example :- Automatic Electric Iron

Control System Terminology

1) **Reference Input** :- It provides Input Signal for desired Output.

2) **Error detector** :- It is an element in which are system variable is subtracted from another variable to obtain third Variable. also called Comparator.

3) **Feedback Element** :- It measures the Controlled output, Convert or transforms to a suitable value for Comparison with reference input.

4) **Error Signal** :- It is an algebraic sum of reference input and feedback.

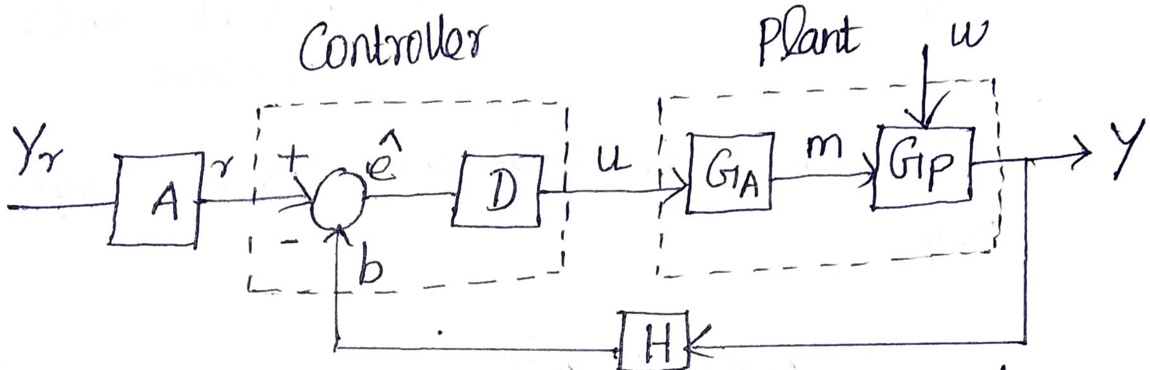
5) **Controller** :- Controller is an element that required to generate the appropriate Control signal. The Controller operates until the error between Controlled output and ~~reduced~~ desired output is reduced to zero.

6) **Controlled System** :- It is a body, a plant, Process or a Machine of which a particular Condition is to be Controlled.

Ex:- Room heating system, Reactor boiler

Controlled Output:- It is produced by actuating signal available as input to the controller. controlled output is made equal to desired output with the help of feedback system.

Basic structure of Control System



$\hat{e} = r - b$

Elements:-

A → Reference input element
Y_r → command input
r → reference input
b → feedback signal
$\hat{e}$ → actuating error
D → Control logic elements
u → Control signal
G_A → Actuator Elements
G_P → Controlled system Elements

m → Manipulated variable
w → Disturbance input
H → feedback elements
w → Disturbance input
y → Controlled output

Open loop (feed forward)	Closed loop (feedback) Control system
No feedback used	feedback is used for Compare the desired output and reference input.
Open loop system is generally Stable	closed loop system become unstable under certain Conditions
Simple to develop and cheap	More complex.
Difficult calibration	Easy calibration
Non linear	linear
Examples- Tv remote, Traffic light	Air Conditioner, Refrigerator

Electrical & Mechanical Transfer function Models.

In Control theory, transfer functions are Commonly used to characterize the input - output relationship of components or systems that can be described by linear time - invariant differential equations.

Transfer function of a Linear time invarient systems is defined as the ratio of Laplace transform of the output to the Laplace transform of the input, under the assumptions that all initial Conditions are zero.

$$\text{Transfer function} = \left.\frac{\text{Laplace transform of output}}{\text{Laplace transform of Input}}\right|_{\text{with zero initial conditions}}$$

Modelling of Mechanical Systems.

It is generally formulated using Newtons law of motion.

Mechanical System

- Translational Motion [Motion takes place in straight lines]
- Rotational Motion [Motion will be in rotation]

Mechanical Translational Systems

The model of mechanical translational systems can be obtained by using three basic elements mass, spring and dash-pot.

When a force is applied to a translational mechanical system, it is opposed by opposing forces due to mass, friction and elasticity of the system.

For translational mechanical systems it states that the sum of applied forces is equal to the sum of opposing forces [Newton's Law].

Input → Force
Output → Displacement

List of Symbols used in Mechanical Translational system

$x =$ displacement, m

$v = \frac{dx}{dt} =$ velocity, m/sec

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} =$ Acceleration, m/sec^2

f = Applied force, N (Newtons)

M = Mass, Kg

K = Stiffness of spring, N/m

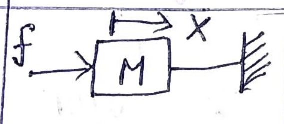
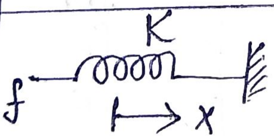
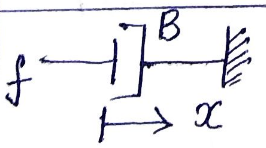
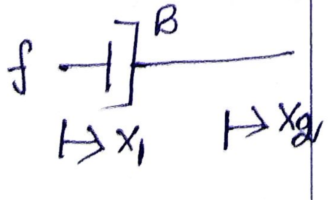
B = Viscous friction co-efficient, N-sec/m

f_m = Opposing force offered by mass of the body, N

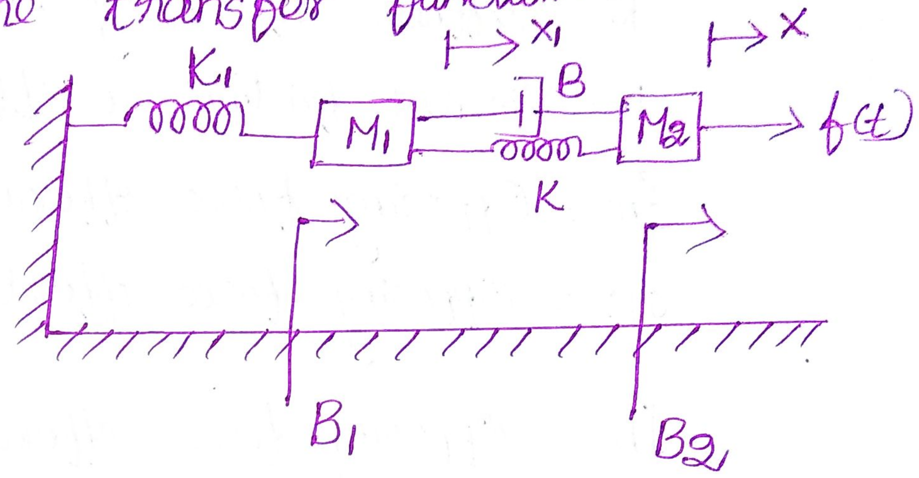
f_K = Opposing force offered by the elasticity of the body (spring), N

f_b = Opposing force offered by the friction of the body (dash-pot), N

Force Balanced equations of Idealized elements

S.No	Element	Symbol	Representation	Proportionality	Time domain	Laplace Transform
1	Mass	M		$M \propto a$	$f_m = M\frac{d^2x}{dt^2}$	$MS^2X(S)$
2	Spring	K		$K \propto x$	$f_K = KX$	$KX(S)$
3	Dash-pot	B		$B \propto V$	$f_b = B\frac{dx}{dt}$	$BSX(S)$
			(or) 	$B \propto (V_1 - V_2)$	$f_b = B\frac{d}{dt}[x_1 - x_2]$	$BS[X_1(S) - X_2(S)]$

1) Write the Differential equation governing the translational Mechanical System as shown in fig & determine the transfer function.

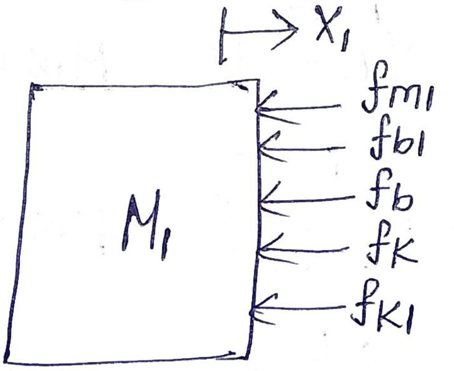


Solution

Transfer Function

$$T.F = \frac{X(S)}{F(S)}$$

Draw the free body diagram of Mass element 'M_1'.



By using Newton's law

Sum of Opposing force = Sum of Applying force

$$f_{m1} + f_{b1} + f_b + f_K + f_{K1} = 0$$

Differential equation

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B \frac{d}{dt}[x_1 - x] + K[x_1 - x] + K_1 x_1 = 0$$

Taking Laplace transform on both sides

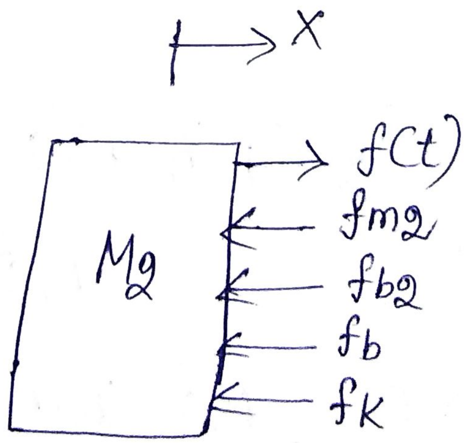
$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs[X_1(s) - X(s)] + K[X_1(s) - X(s)]$$
$$+ K_1 X_1(s) = 0.$$

$$X_1(s)[M_1 s^2 + B_1 s + Bs + K + K_1] - X(s)[Bs + K] = 0.$$

$$X_1(s)[M_1 s^2 + B_1 s + Bs + K + K_1] = X(s)[Bs + K]$$

$$\boxed{X_1(s) = \frac{X(s)[Bs + K]}{M_1 s^2 + (B_1 + B)s + (K + K_1)}}$$

Draw the free body diagram of Mass element M_2



By Newton's law

$$f(t) = f_{m2} + f_{b2} + f_b + f_K$$

$$f(t) = M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}[x - x_1] + K[x - x_1]$$

Taking Laplace transform on both sides

$$F[S] = M_2 S^2 X(S) + B_2 S X(S) + BS[X(S) - X_1(S)] + K[X(S) - X_1(S)]$$

$$F(S) = [M_2 S^2 + B_2 S + BS + K] X(S) - (BS+K) X_1(S)$$

Substitute $X_1(S)$ value to above

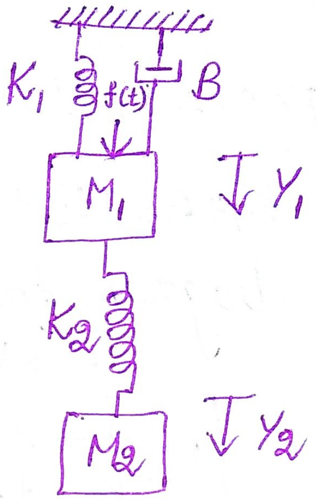
$$F(S) = [M_2 S^2 + B_2 S + BS + K] X(S) - \frac{[BS+K][BS+K]X(S)}{M_1 S^2 + (K_1+K) + (B_1+B)S}$$

$$F(S) = \left[(M_2 S^2 + B_2 S + BS + K) - \frac{(BS+K)^2}{M_1 S^2 + (B_1+B)S + K_1 + K}\right] X(S)$$

$$\frac{F(S)}{X(S)} = \frac{(M_2 S^2 + (B_2+B)S + K)(M_1 S^2 + (B_1+B)S + (K_1+K)) - (BS+K)^2}{M_1 S^2 + (B_1+B)S + K_1 + K}$$

$$\boxed{T.F = \frac{X(S)}{F(S)} = \frac{M_1 S^2 + (B_1+B)S + K_1 + K}{[M_2 S^2 + (B_2+B)S + K](M_1 S^2 + (B_1+B)S + (K_1+K)) - (BS+K)^2}}$$

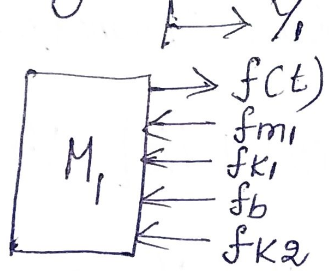
Determine Transfer function for the given diagram



Solution

$$T.F = \frac{Y_2(s)}{F(s)}$$

The free body diagram of mass 'M_1'



By Newton's law

$$f(t) = f_{K1} + f_b + f_{K2} + f_{m1}$$

$$f(t) = K_1 Y_1 + B\frac{dY_1}{dt} + K_2[Y_1 - Y_2] + M_1\frac{d^2Y_1}{dt^2}$$

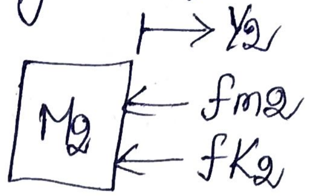
Apply laplace transform on both sides

$$F(s) = K_1 Y_1(s) + B s Y_1(s) + K_2[Y_1(s) - Y_2(s)] + M_1 s^2 Y_1(s)$$

$$F(S) = [K_1 + BS + K_2 + M_1 S^2]\, Y_1(S) - K_2 Y_2(S)$$

$$Y_1(S) = \frac{F(S) + K_2 Y_2(S)}{K_1 + BS + K_2 + M_1 S^2} \longrightarrow (1)$$

Free Body diagram for Mass M_2



By Newton's law

$$f_{m2} + f_{K2} = 0.$$

$$M_2 \frac{d^2 y_2}{dt^2} + K_2 [y_2 - y_1] = 0$$

Apply laplace Transform

$$M_2 S^2 Y_2(S) + K_2 [Y_2(S) - Y_1(S)] = 0$$

$$[M_2 S^2 + K_2]\, Y_2(S) - K_2 Y_1(S) = 0$$

$$[M_2 S^2 + K_2]\, Y_2(S) - K_2 \left[\frac{F(S) + K_2 Y_2(S)}{K_1 + BS + K_2 + M_1 S^2}\right] = 0.$$

$$Y_2(S) \left[M_2 S^2 + K_2 - \frac{K_2^2}{K_1 + BS + K_2 + M_1 S^2} \right] = \frac{K_2 F(S)}{M_1 S^2 + BS + (K_1 + K_2)}$$

$$Y_2(S)\left[\frac{(M_2S^2+K_2)\,(M_1S^2+BS+K_1+K_2)-K_2^2}{M_1S^2+BS+(K_1+K_2)}\right]=\frac{K_2F(S)}{M_1S^2+BS+(K_1+K_2)}$$

$$\boxed{T.F=\frac{Y_2(S)}{F(S)}=\frac{K_2}{\left[(M_1S^2+BS+(K_1+K_2))\,(M_2S^2+K_2)-K_2^2\right]}}$$

Mechanical Rotational Systems

The model of rotational mechanical systems can be obtained by using three elements, moment of inertia [J] of mass, dash-pot with rotational frictional coefficient [B] and torsional spring with stiffness [K].

Input → Torque

Output → Angular Displacement

When a torque is applied to a rotational mechanical system, it is opposed by opposing torques due to moment of inertia, friction and elasticity of the system.

List of symbols used in Mechanical Rotational System

θ = Angular displacement, rad

$\frac{d\theta}{dt}$ = Angular velocity, rad/sec

$\frac{d^2\theta}{dt^2}$ = Angular Acceleration, rad/sec^2

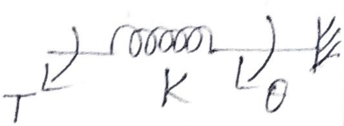
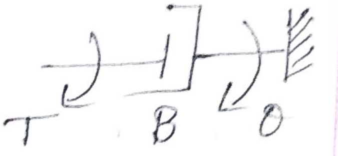
T = Applied torque - N-m

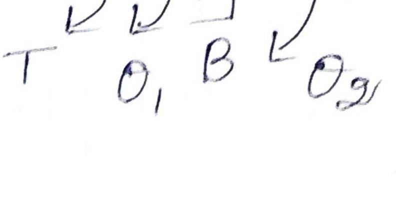
J - Moment of inertia, $Kg\text{-}m^2/rad$

B - Rotational frictional coefficient, $N\text{-}m/(rad/sec)$

K - Stiffness of the spring, $N\text{-}m/rad$

Torque Balance Equations of Idealised Elements

S. No	Element	Symbol	Representation	Time domain	Laplace Transform
1	Moment of inertia	J		$T_j = J\frac{d^2\theta}{dt^2}$	$J s^2 \theta(s)$
2	Torsional Spring	K		$T_K = K\theta$	$K\theta(s)$
3	dash-pot	B		$T_b = B\frac{d\theta}{dt}$	$B s \theta(s)$

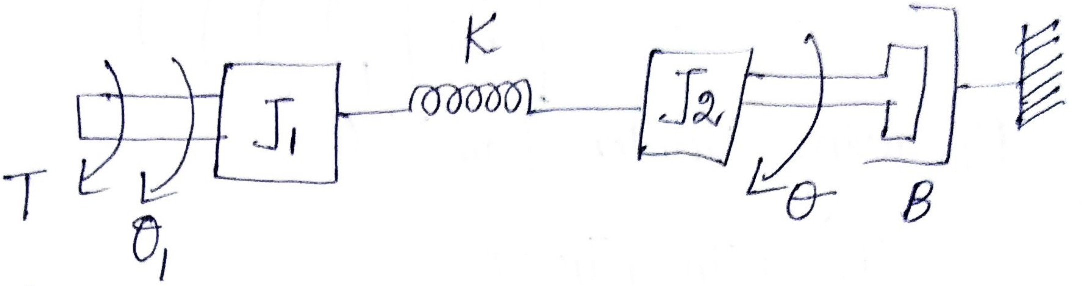


$$T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

Laplace Transform

$$Bs[\theta_1(s) - \theta_2(s)]$$

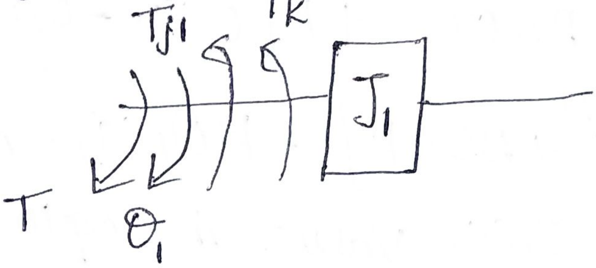
1) Determine Transfer function for the following Mechanical Rotational System.



Solution

$$T.F = \frac{\theta(s)}{T(s)}$$

Free Body diagram of moment of inertia element J_1



$$T_{J1} + T_K = T$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T \longrightarrow ①$$

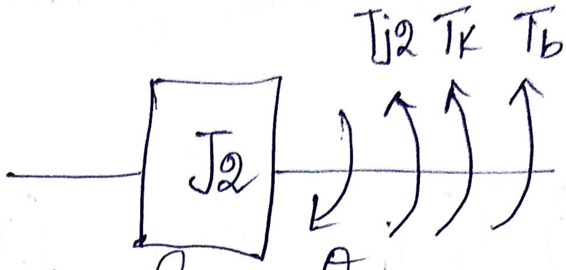
Taking Laplace Transform on both sides.

$$J_1 s^2 \theta_1(s) + K[\theta_1(s) - \theta(s)] = T(s)$$

$$\theta_1(s)[J_1 s^2 + K] - K\theta(s) = T(s)$$

$$\theta_1(s) = \frac{T(s) + K\theta(s)}{J_1 s^2 + K} \longrightarrow ②$$

free body diagram of moment of inertia element J2.



By using Newtons law

$$T_{j2} + T_B + T_K = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0 \rightarrow (3)$$

Taking Laplace transform on both sides

$$J_2 s^2 \theta(s) + Bs\theta(s) + K[\theta(s) - \theta_1(s)] = 0.$$

$$\theta(s)[J_2 s^2 + Bs + K] - K\theta_1(s) = 0. \rightarrow (4)$$

Substituting $\theta_1(s)$ value in eqn (4)

$$\theta(s)[J_2 s^2 + Bs + K] - K\left[\frac{T(s) + K\theta(s)}{J_1 s^2 + K}\right] = 0.$$

$$\theta(s)\left[J_2 s^2 + Bs + K - \frac{K^2}{J_1 s^2 + K}\right] = \frac{K\,T(s)}{J_1 s^2 + K}$$

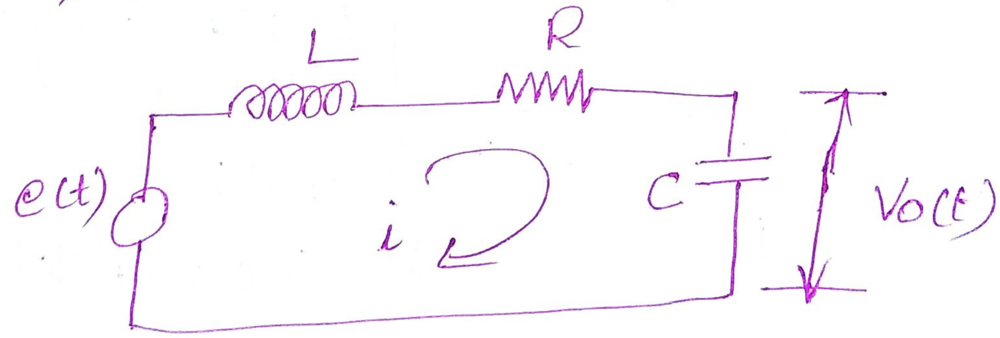
$$\boxed{T.F = \frac{\theta(s)}{T(s)} = \frac{K}{(J_2 s^2 + Bs + K)(J_1 s^2 + K) - K^2}}$$

Modelling of Electrical System.

Resistor, Inductor, capacitor are the three basic elements of an electric circuit. The circuit is analysed by application of Kirchoff Voltage and current laws.

S.No	Element	Symbol	Time domain $v(t)$	Laplace Transform $V(s)$
1	Resistor	$i(t)$, R, $+\ v(t)\ -$	$i(t) \times R$	$R\,I(s)$
2	Inductor	$i(t)$, L, $+\ v(t)\ -$	$L \frac{d}{dt} i(t)$	$LS\,I(S)$
3	Capacitor	$i(t)$, C, $+\ v(t)\ -$	$\frac{1}{C}\int i(t)\,dt$	$\frac{1}{SC} I(S)$

1) Consider an electrical system of RLC series circuit, Find out the transfer function:



Solution

Apply KVL

$$e(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt \rightarrow (1)$$

Taking Laplace Transform on both sides

$$E(S) = LSI(S) + RI(S) + \frac{1}{CS} I(S)$$

$$E(S) = \left[LS + R + \frac{1}{CS}\right] I(S)$$

$$E(S) = \left[\frac{LCS^2 + RCS + 1}{CS}\right] I(S) \rightarrow (2)$$

Let the output voltage $V_o(t)$ be taken across the capacitor C then

$$V_o(t) = \frac{1}{C}\int i\,dt \rightarrow (3)$$

Taking Laplace Transform on both sides.

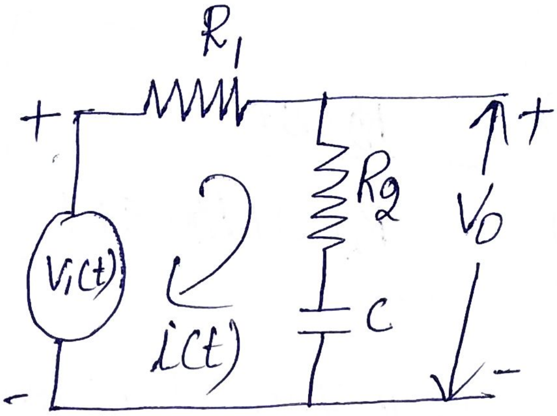
$$V_o(S) = \frac{1}{CS} I(S) \rightarrow (4)$$

$$T.F = \frac{V_o(S)}{E(S)} = \frac{I(S)/CS}{\left[\frac{LCS^2 + RCS + 1}{CS}\right] I(S)}$$

$$\frac{V_o(S)}{E(S)} = \frac{I(S)}{CS} \times \frac{CS}{(LCS^2 + RCS + 1)\, I(S)}$$

$$\boxed{\frac{V_o(S)}{E(S)} = \frac{1}{LCS^2 + RCS + 1}}$$

2) Determine the transfer function of electrical system as shown in figure



Apply KVL

$$V_i(t) = R_1 i(t) + R_2 i(t) + \frac{1}{C}\int i(t)\, dt \rightarrow (1)$$

Taking Laplace Transform on both sides.

$$V_i(s) = R_1 I(s) + R_2 I(s) + \frac{1}{Cs} I(s)$$

$$V_i(s) = \left[R_1 + R_2 + \frac{1}{Cs}\right] I(s) \rightarrow (2)$$

Output equation

$$V_o(t) = R_2 i(t) + \frac{1}{C}\int i(t)\, dt \rightarrow (3)$$

Taking Laplace transform on both sides

$$V_o(s) = R_2 I(s) + \frac{1}{Cs} I(s)$$

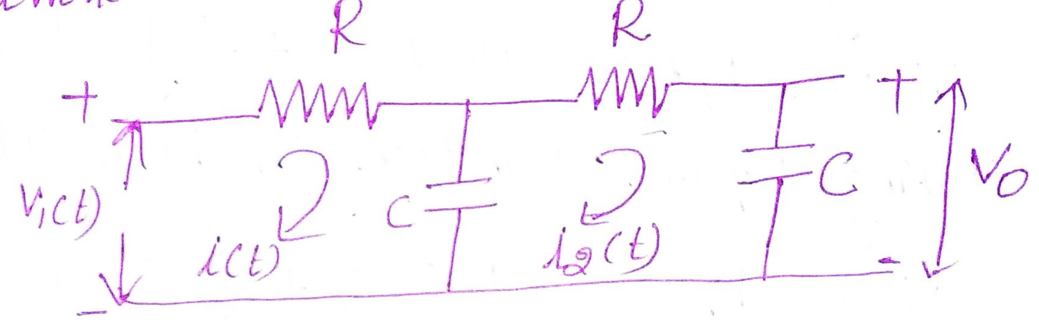
$$V_o(s) = \left[R_2 + \frac{1}{Cs}\right] I(s) \rightarrow (4)$$

$$T.F = \frac{V_o(s)}{V_i(s)} = \frac{\left[R_2 + \frac{1}{Cs}\right] I(s)}{\left[R_1 + R_2 + \frac{1}{Cs}\right] I(s)}$$

$$\frac{V_o(S)}{V_i(S)} = \frac{R_2(CS+1)}{CS} \times \frac{CS}{R_1CS+R_2CS+1}$$

$$\boxed{\frac{V_o(S)}{V_i(S)} = \frac{R_2CS+1}{R_1CS+R_2CS+1}}$$

3) Obtain the transfer function for electrical Network



Solution

Apply KVL to loop 1

$$V_i(t) = R\,i(t) + \frac{1}{C}\int\left[i_1(t) - i_2(t)\right] dt$$

Taking Laplace Transform on both sides

$$V_i(S) = RI(S) + \frac{1}{CS}\left[I_1(S) - I_2(S)\right]$$

$$V_i(S) = RI(S) + \frac{1}{CS}I_1(S) - \frac{1}{CS}I_2(S)$$

$$V_i(S) + \frac{1}{CS}I_2(S) = \left[R + \frac{1}{CS}\right]I_1(S)$$

$$I_1(S) = \frac{V_i(S)\,CS + I_2(S)}{CS} \times \frac{CS}{1+RCS}$$

$$\boxed{I_1(S) = \frac{V_i(S)\,CS + I_2(S)}{1+RCS}}$$

Apply KVL to loop 2

$$R\,i_2(t) + \frac{1}{C}\int i_2(t)\,dt + \frac{1}{C}\int\left[i_2(t) - i_1(t)\right]dt = 0.$$

Taking Laplace Transform on both sides

$$R\,I_2(S) + \frac{1}{CS}I_2(S) + \frac{1}{CS}\left[I_2(S) - I_1(S)\right] = 0$$

$$I_2(S)\left[R + \frac{1}{CS} + \frac{1}{CS}\right] = \frac{1}{CS}I_1(S)$$

Substitute $I_1(S) = \frac{V_i(S)\,CS + I_2(S)}{1+RCS}$

$$I_2(S)\left[\frac{2+RCS}{CS}\right] = \frac{1}{CS}\left[\frac{V_i(S)\cdot CS + I_2(S)}{1+RCS}\right]$$

$$I_2(S)\left[\frac{2+RCS}{CS}\right] = \frac{V_i(S)}{1+RCS} + \frac{I_2(S)}{CS[1+RCS]}$$

$$I_2(S)\left[\frac{2+RCS}{CS} - \frac{1}{CS[1+RCS]}\right] = \frac{V_i(S)}{1+RCS}$$

$$I_2(S)\left[\frac{(2+RCS)(1+RCS)-1}{CS[1+RCS]}\right] = \frac{V_i(S)}{1+RCS}$$

$$I_2(s) = \frac{V_i(s)\,Cs}{1+3RCs+R^2C^2s^2}$$

$$\boxed{V_i(s) = \frac{I_2(s)}{Cs}\left[1+3RCs+R^2C^2s^2\right]}$$

Output equation $V_o(t) = \frac{1}{C}\int i_2(t)\,dt$

Taking Laplace transform

$$V_o(s) = \frac{1}{Cs} I_2(s)$$

$$T.F = \frac{V_o(s)}{V_i(s)} = \frac{I_2(s)}{Cs} \times \frac{Cs}{I_2(s)\left[1+3RCs+R^2C^2s^2\right]}$$

$$\boxed{T.F = \frac{V_o(s)}{V_i(s)} = \frac{1}{1+3RCs+R^2C^2s^2}}$$

Analogous System

Sometimes Mechanical and other Systems are converted into electrical analogous systems for the easy of design, modification and analysis. Analogous systems have same type of differential equations.

There are four types of analogies namely.

(i) Force - Voltage Analogy
(ii) Force - Current Analogy
(iii) Torque Voltage Analogy
(iv) Torque Current Analogy

The translational Mechanical system is represented by mass, spring and dash-pot in equation

$$\boxed{M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f(t)}$$

The Rotational Mechanical System is represented by

$$\boxed{J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = T}$$

The electrical system is represented by Inductor, Resistor, capacitor interms of voltage

$$V(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i\,dt$$

But $\boxed{i(t) = \frac{dq}{dt}}$ (ie). current is rate of flow of charge

$$\boxed{V(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q}$$

The electrical system can be represented by capacitor, Resistor, inductor in terms of current.

$$i(t) = \frac{1}{L}\int e\,dt + C\frac{de}{dt} + \frac{1}{R}e$$

e (OR) v(t)

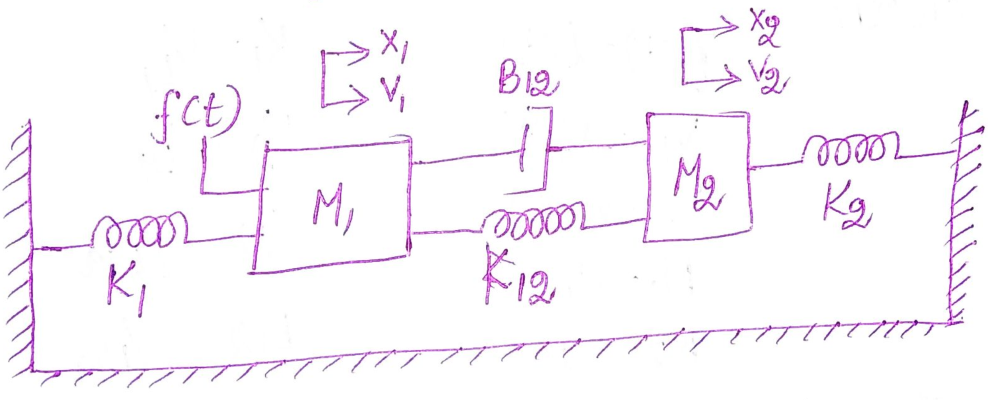
But $e = \frac{d\psi}{dt}$ $\psi \Rightarrow$ flux linkage.

$$\boxed{i(t) = C\frac{d^2\psi}{dt^2} + \frac{1}{R}\frac{d\psi}{dt} + \frac{1}{L}\psi}$$

Compare the above equations, which gives equivalent parameters. It is given by table.

S.No	Translational	Rotational	Electrical system interm of Voltage	Electrical system interm of I
1	F	T	$v(t)$	$i(t)$
2	M	J	L	C
3	B	B	R	$1/R$
4	K	K	$1/C$	$1/L$
5	x	θ	q	ψ
6	V (or) $\frac{dx}{dt}$	ω (or) $\frac{d\theta}{dt}$	$i(t)$	$v(t)$

1) Draw the force voltage & force current analogy for the following system as shown in fig.



Step 1

Identify equivalent electrical systems in terms of voltage

$f(t) = v(t)$

$K_1 = 1/C_1$

$M_1 = L_1$

$V_1 \rightarrow i_1(t)$

$B_{12} = R_{12}$

$K_{12} = 1/C_{12}$

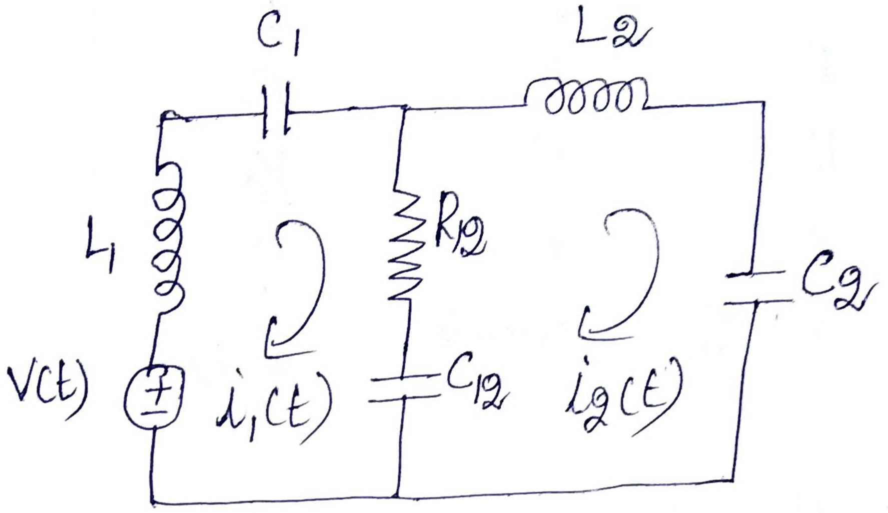
$M_2 = L_2$

$V_2 = i_2(t)$

$K_2 = 1/C_2$

Step 2:-

Force-Voltage circuit diagram



Step 3:-

Identify equivalent electrical term interms of current

$f(t) = i(t)$

$V_1 = V_1(t)$

$V_2 = V_2(t)$

$M_1 = C_1$

$M_2 = C_2$

$K_1 = 1/L_1$

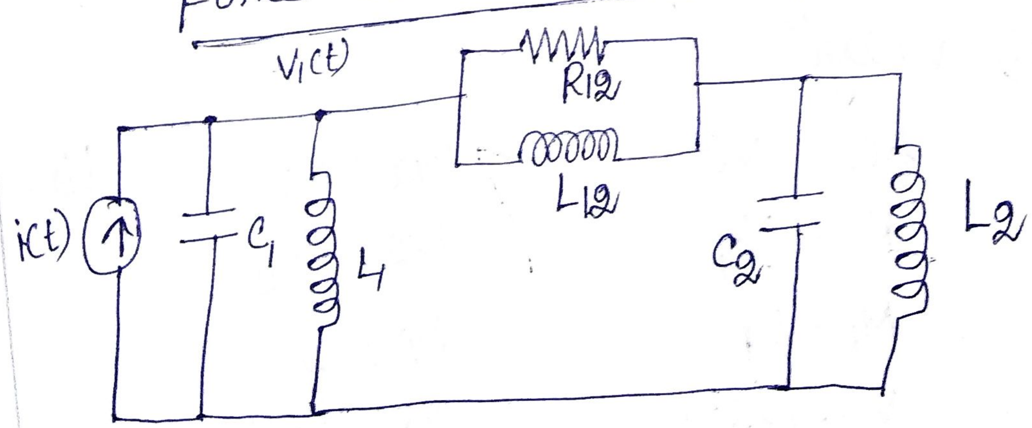
$K_2 = 1/L_2$

$K_{12} = 1/L_{12}$

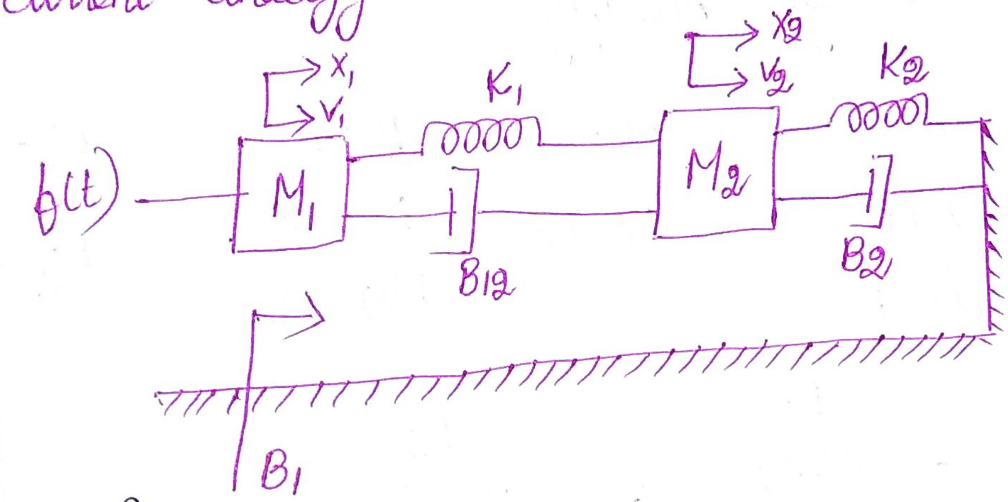
$B_{12} = 1/R_{12}$

Step 4:-

Force-current Circuit Diagram

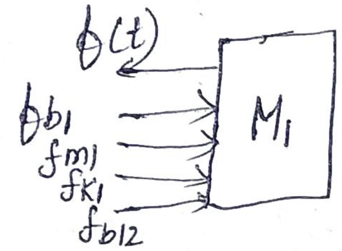


2) Write a differential equation for mechanical system as shown in figure. Draw force voltage & force current analogy circuit.



Solution

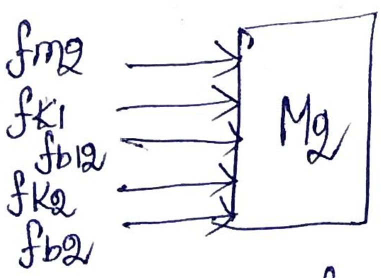
Draw free body diagram of M_1



$$f(t) = f_{m1} + f_{K1} + f_{b12} + f_{b1}$$

$$f(t) = M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_1 (x_1 - x_2) + B_{12} \frac{d}{dt}(x_1 - x_2)$$

Draw free body diagram of M_2



$$f_{m2} + f_{K2} + f_{b2} + f_{K1} + f_{b12} = 0$$

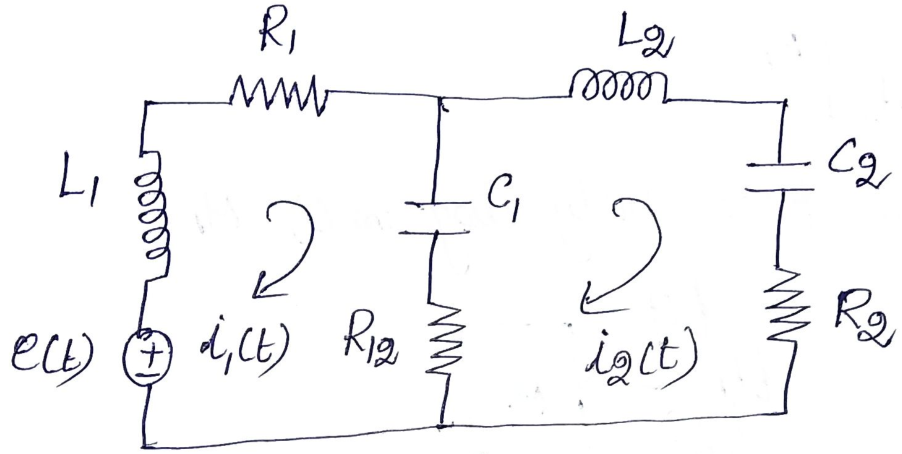
$$M_2 \frac{d^2x_2}{dt^2} + K_2 x_2 + B_2 \frac{dx_2}{dt} + K_1(x_2 - x_1) + B_{12}\frac{d}{dt}(x_2 - x_1) = 0$$

Force Voltage Analogy

$f(t) = e(t)$ $\quad M_1 = L_1$ $\quad B_1 = R_1$ $\quad K_1 = 1/C_1$

$V_1 = i_1(t)$ $\quad M_2 = L_2$ $\quad B_2 = R_2$ $\quad K_2 = 1/C_2$

$V_2 = i_2(t)$ $\quad B_{12} = R_{12}$

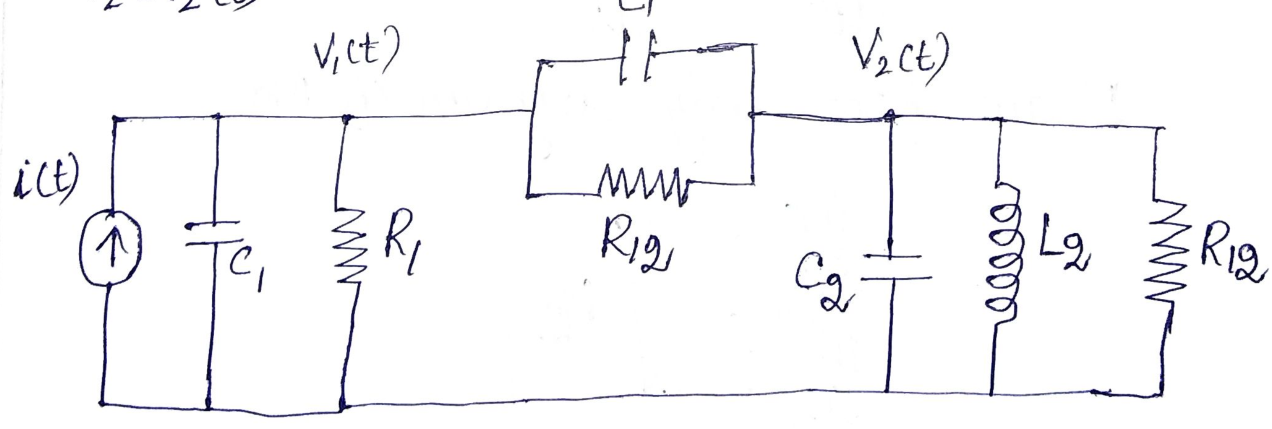


Force Current Analogy System

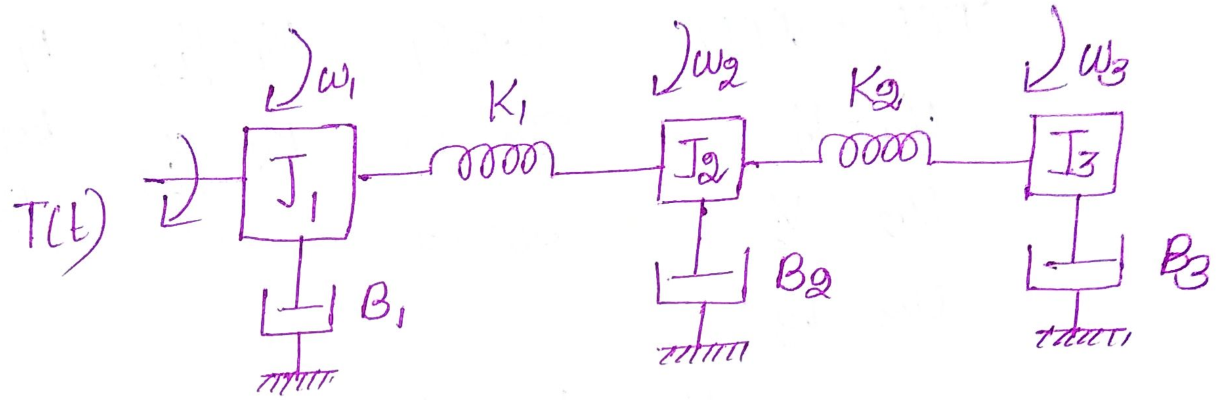
$f(t) = i(t)$ $\quad M_1 = C_1$ $\quad K_1 = 1/L_1$ $\quad B_1 = 1/R_1$

$V_1 = V_1(t)$ $\quad M_2 = C_2$ $\quad K_2 = 1/L_2$ $\quad B_2 = 1/R_2$

$V_2 = V_2(t)$ $\quad B_{12} = 1/R_{12}$



3) Draw the Torque Voltage & Torque Current analogous system for the Mechanical System.



Step 1:-

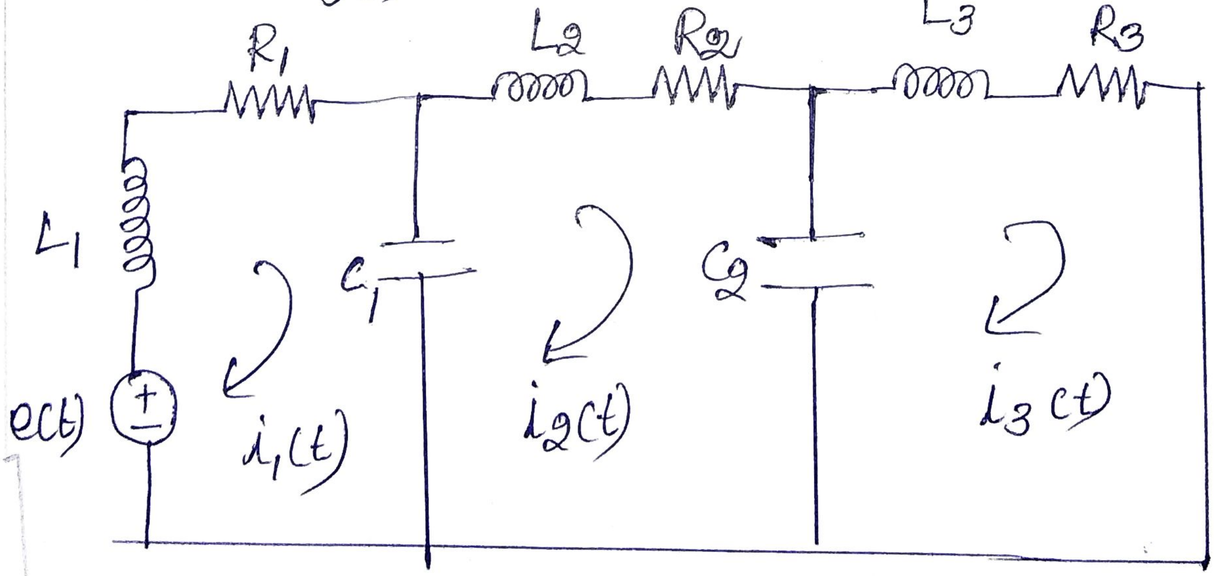
Identify equivalent electrical systems interms of voltage.

$T(t) = e(t)$ $\quad J_1 = L_1$ $\quad K_1 = 1/C_1$ $\quad B_1 = R_1$

$\omega_1 = i_1(t)$ $\quad J_2 = L_2$ $\quad K_2 = 1/C_2$ $\quad B_2 = R_2$

$\omega_2 = i_2(t)$ $\quad J_3 = L_3$ $\quad B_3 = R_3$

$\omega_3 = i_3(t)$



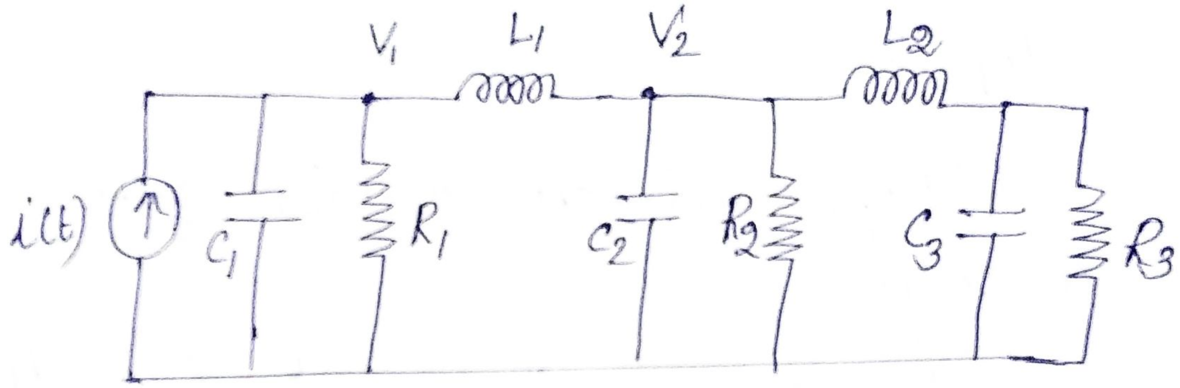
Identify equivalent electrical system interms of current.

$T(t) = i(t)$ $\qquad J_1 = C_1$ $\qquad K_1 = 1/L_1$ $\qquad B_1 = 1/R_1$

$\omega_1 = V_1$ $\qquad J_2 = C_2$ $\qquad K_2 = 1/L_2$ $\qquad B_2 = 1/R_2$

$\omega_2 = V_2$ $\qquad J_3 = C_3$ $\qquad B_3 = 1/R_3$

$\omega_3 = V_3$

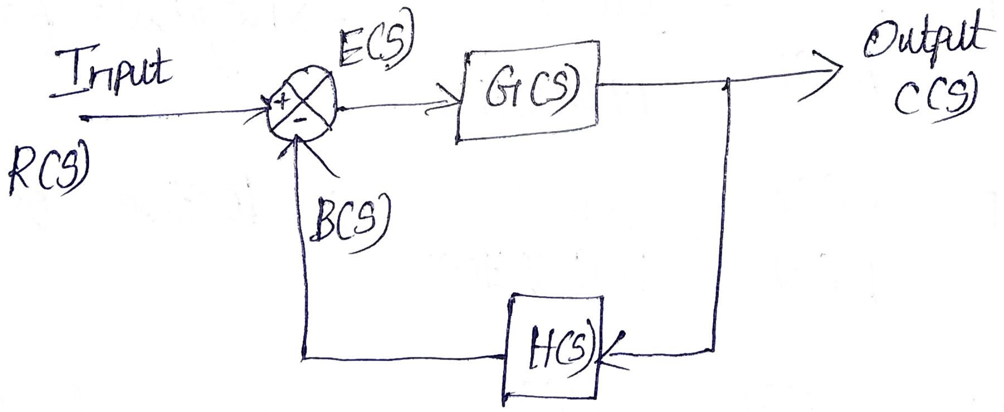


Block Diagram Reduction Technique

Any control system will have a Number of Control Components. A Control system can be represented in Block diagram form. The arrow head pointing towards a particular block indicates input to the system. The arrow head leading away from the block indicates output

Block diagram is possible to evaluate the contribution of each of Components towards overall performance of Control system.

Block diagram representation of closed loop



Block diagram is also called canonical form

$$E(S) = R(S) - B(S)$$

$$B(S) = H(S)\ C(S)$$

$$\boxed{C(S) = E(S)\ G(S)}$$

$$R(s) = E(s) + B(s)$$
$$= E(s) + H(s)\, C(s)$$
$$= E(s) + H(s)\, E(s)\, G(s)$$
$$\boxed{R(s) = E(s) + H(s)\, E(s)\, G(s)}$$

$\therefore$ Closed loop T.F $\Big\}$ $\dfrac{C(s)}{R(s)} = \dfrac{E(s)\, G(s)}{E(s) + H(s)\, E(s)\, G(s)}$

$\div$ by $E(s)\, G(s)$ on N.r & D.r

$$\frac{C(s)}{R(s)} = \frac{1}{\frac{1}{G(s)} + H(s)}$$

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$ → Negative feedback

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)H(s)}$$ → Positive feedback

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)}$$ → unity feedback ($H(s) = 1$)

Rules for Block diagram Simplification

There are some rules which helps to simplify a block diagram of control system. and there are 3 basic elements in Block diagram.

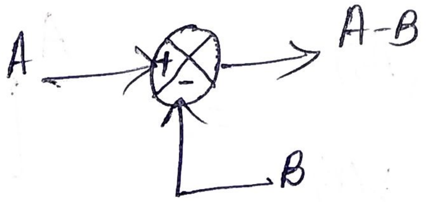
Block:

It is a symbol for mathematical operation on the input signal to the block that produces the output.

$\boxed{B = AG(S)}$

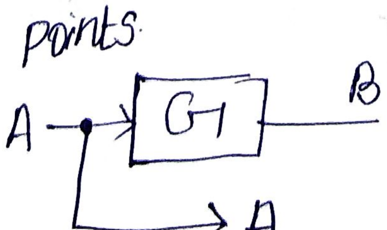
Summing Point

It is used to add two or more signals in the system.



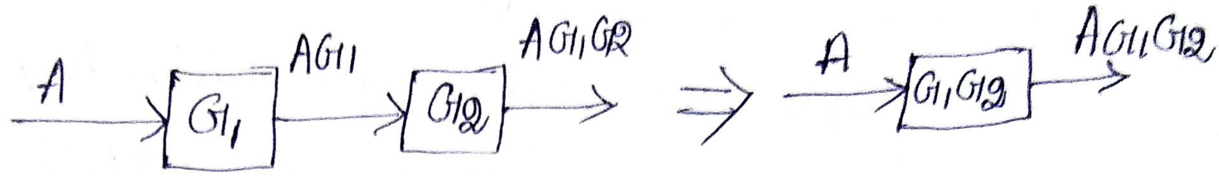
Branch Point

It is a point from which the signal from a block goes concurrently to other blocks or summing points.

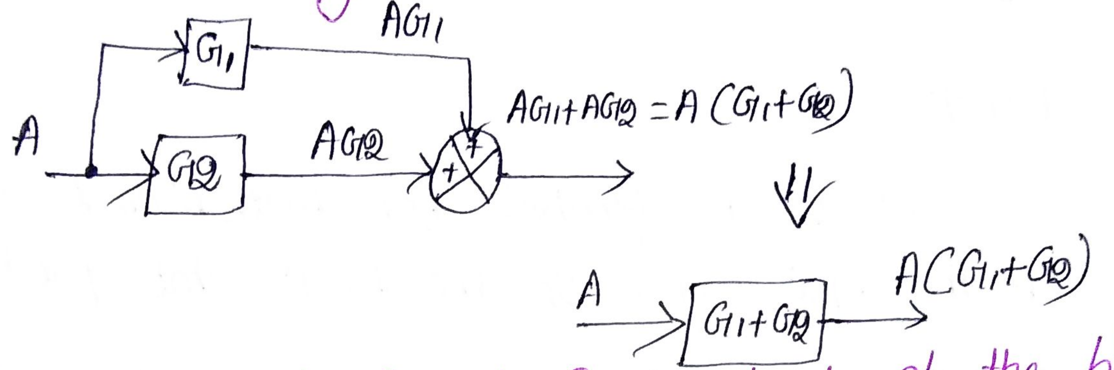


$B = AG_1$

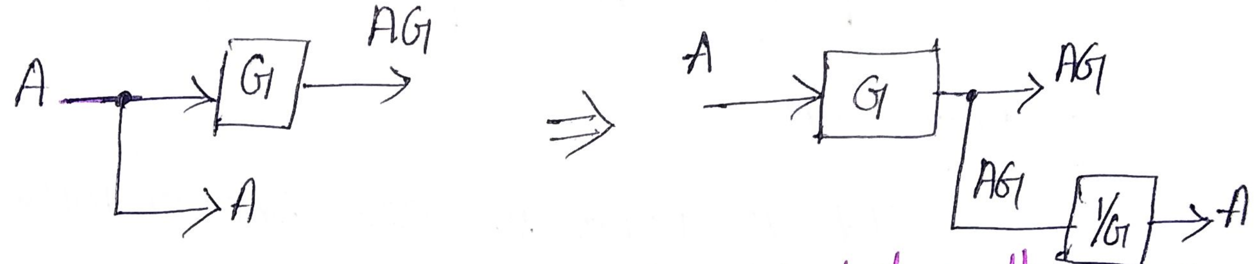
Rule 1:- Combining the blocks in cascade



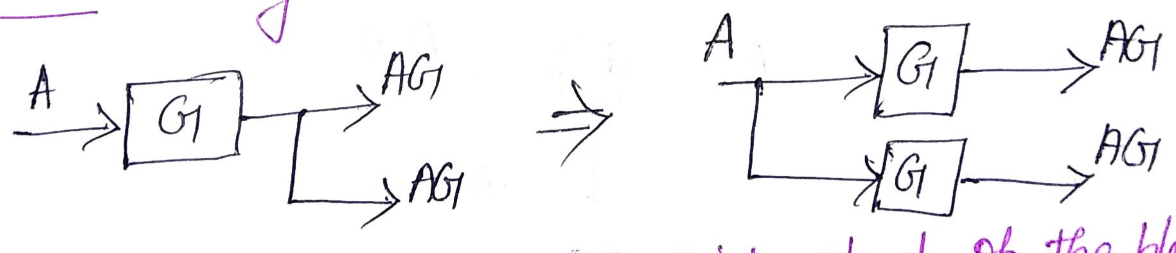
Rule 2:- Combining Parallel blocks



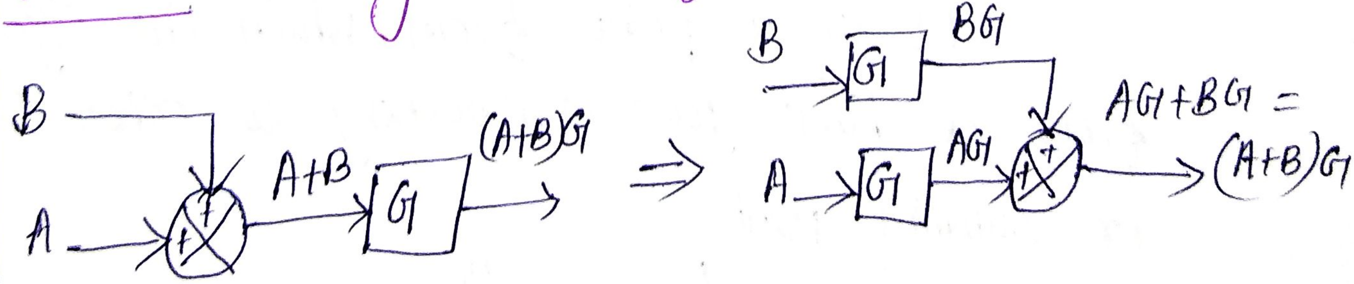
Rule 3:- Moving the Branch Point ahead of the block



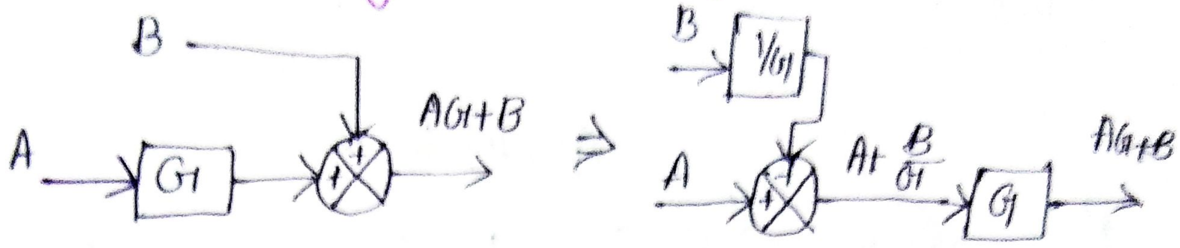
Rule 4:- Moving the Branch point before the block



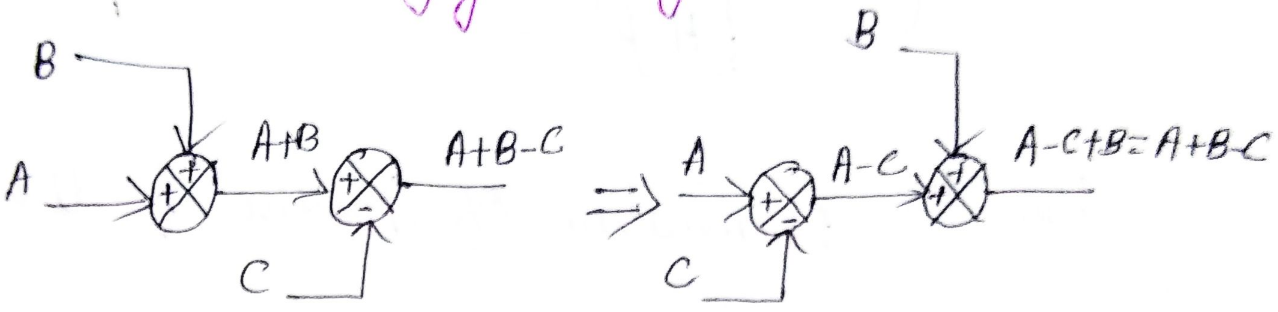
Rule 5:- Moving the summing point ahead of the block



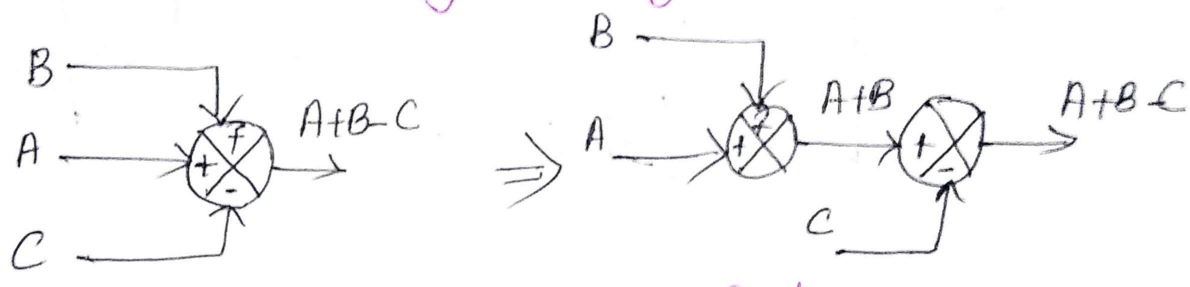
Rule 6:- Moving the summing point before the block



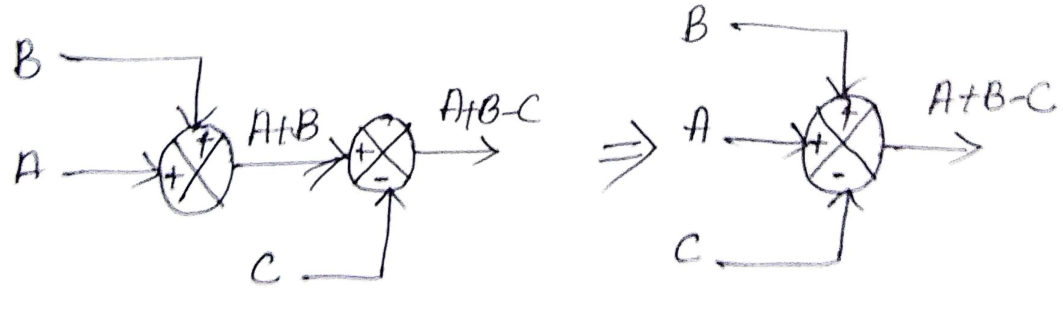
Rule 7:- Interchanging Summing Point



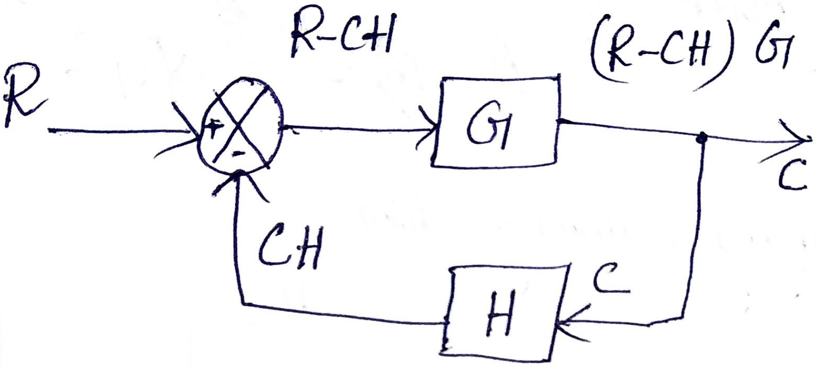
Rule 8: Splitting summing points



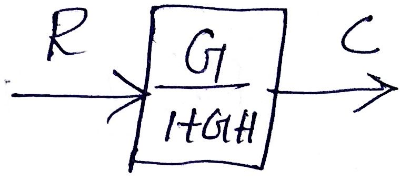
Rule 9:- Combining Summing Points



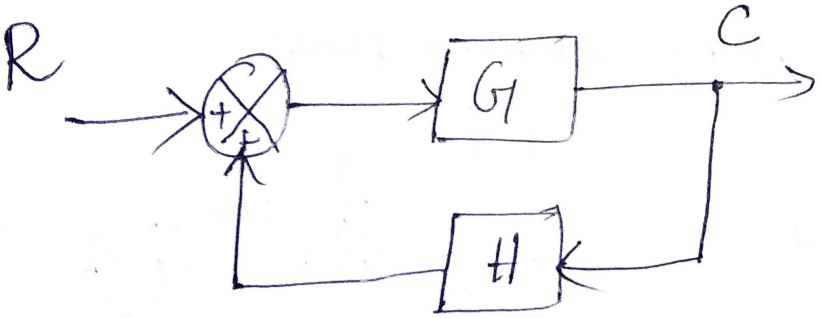
Rule 10:- Elimination of (negative) feedback loop



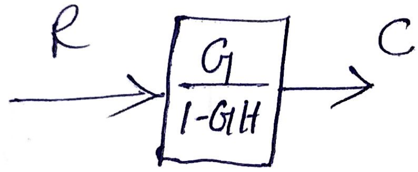
$\Rightarrow$



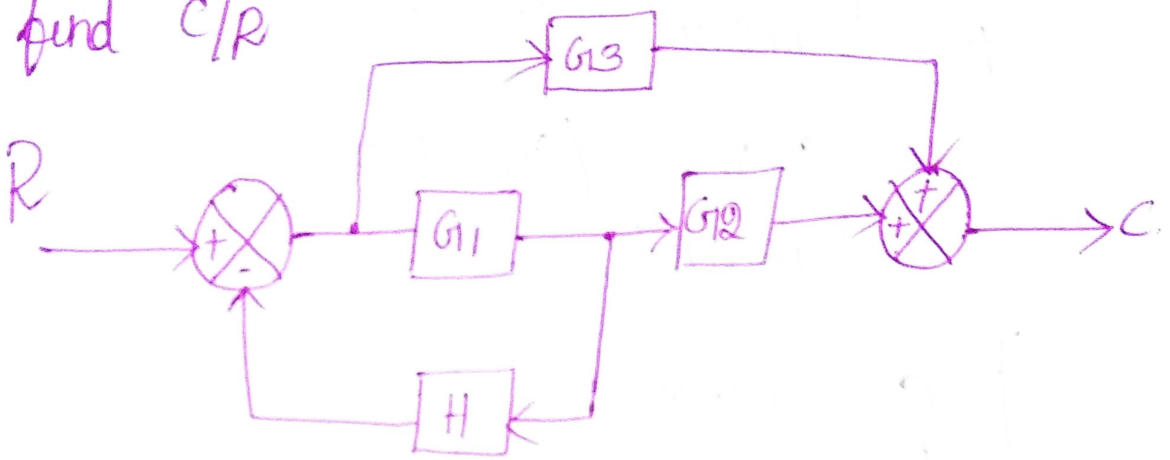
Rule 11:- Elimination of (positive) feedback loop



$\Rightarrow$

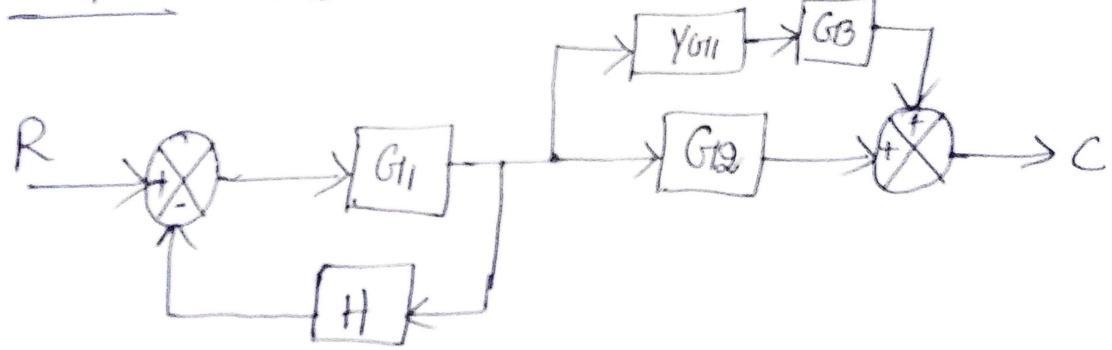


1) Reduce the block diagram as shown in fig and find C/R

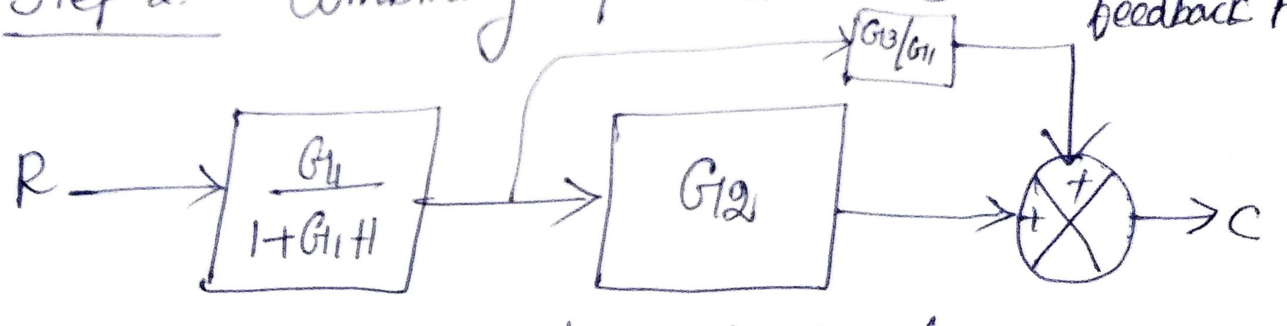


Solution

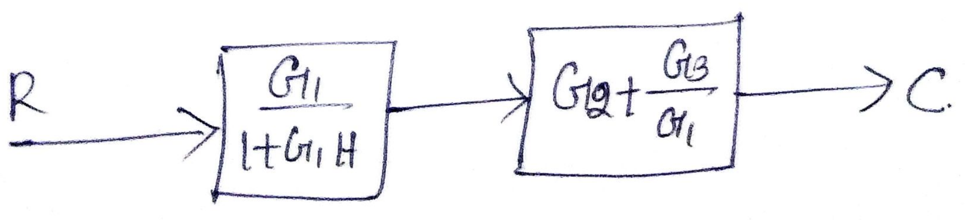
Step 1 :- Move the Branch point after the block.



Step 2:- Combining parallel blocks & Eliminating feedback path



Step 3:- Combining blocks in Cascade

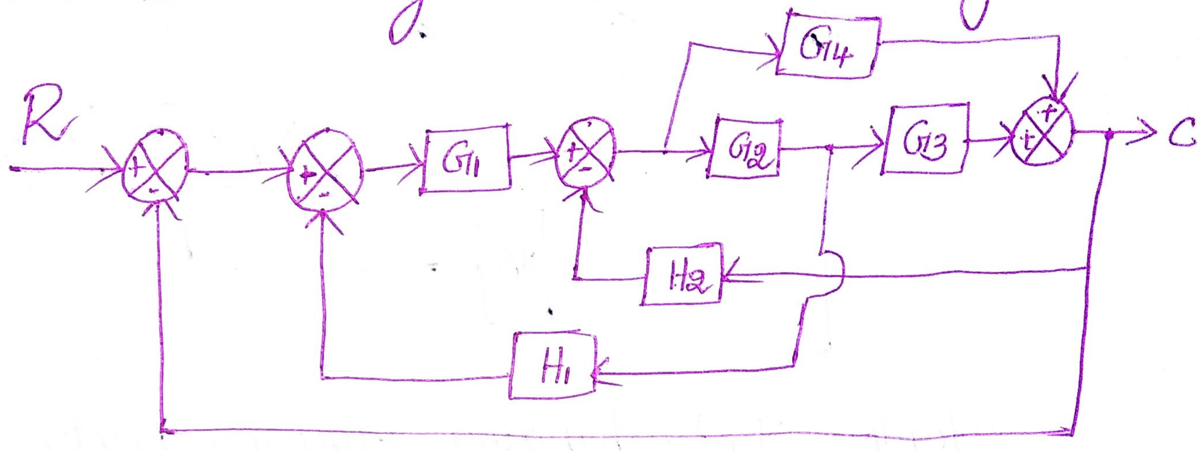


$$\frac{C}{R} = \left(\frac{G_1}{1+G_1H}\right)\left(G_2 + \frac{G_3}{G_1}\right)$$

$$= \left(\frac{G_1}{1+G_1H}\right)\left(\frac{G_2G_1 + G_3}{G_1}\right)$$

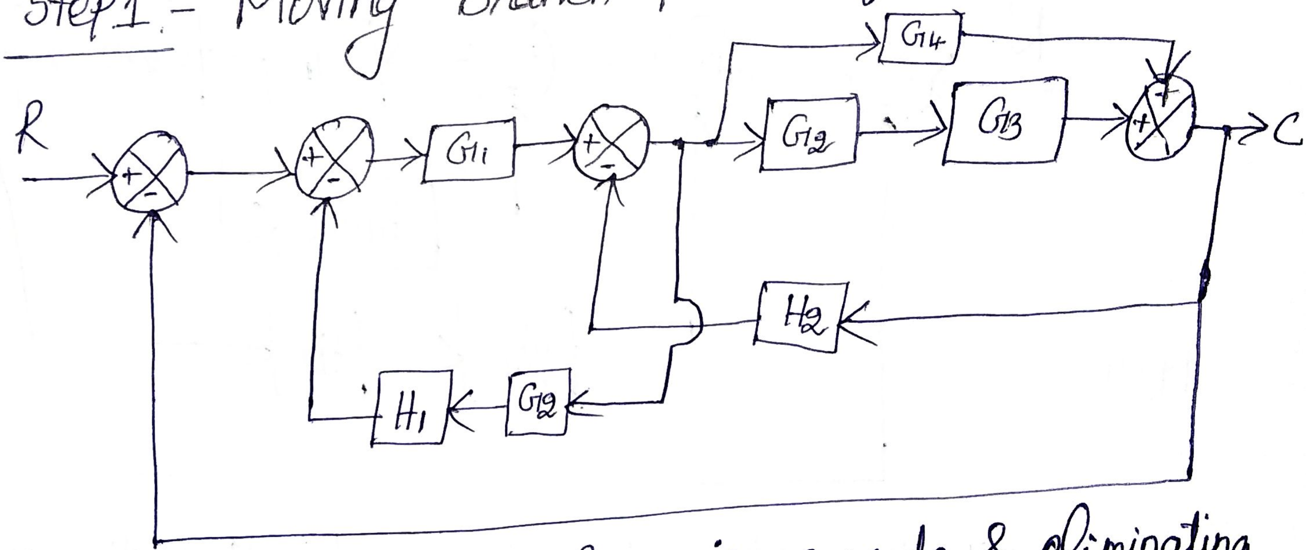
$$\boxed{\frac{C}{R} = \frac{G_1G_2 + G_3}{1+G_1H}}$$

Using Block diagram reduction technique find closed loop transfer function of the system whose Block diagram is shown in figure.

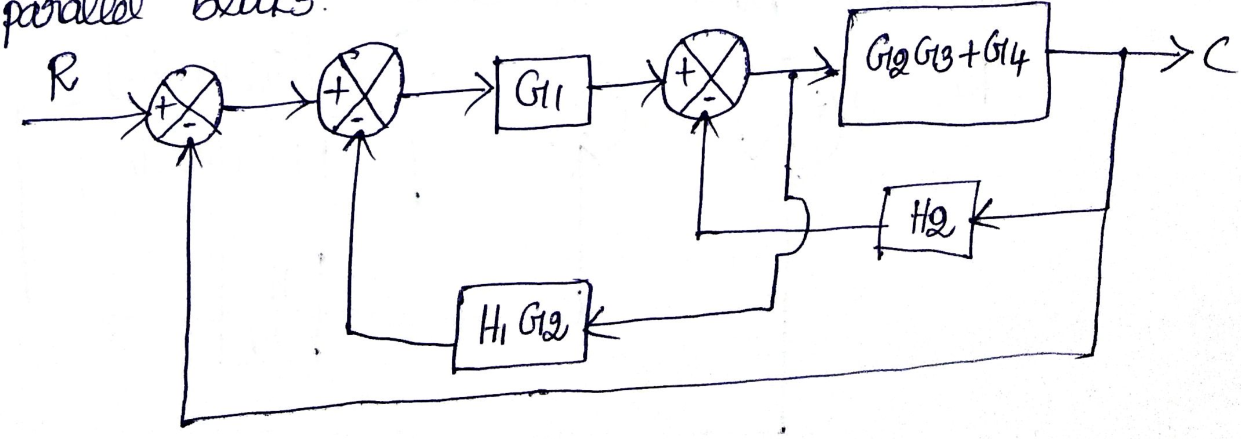


Solution

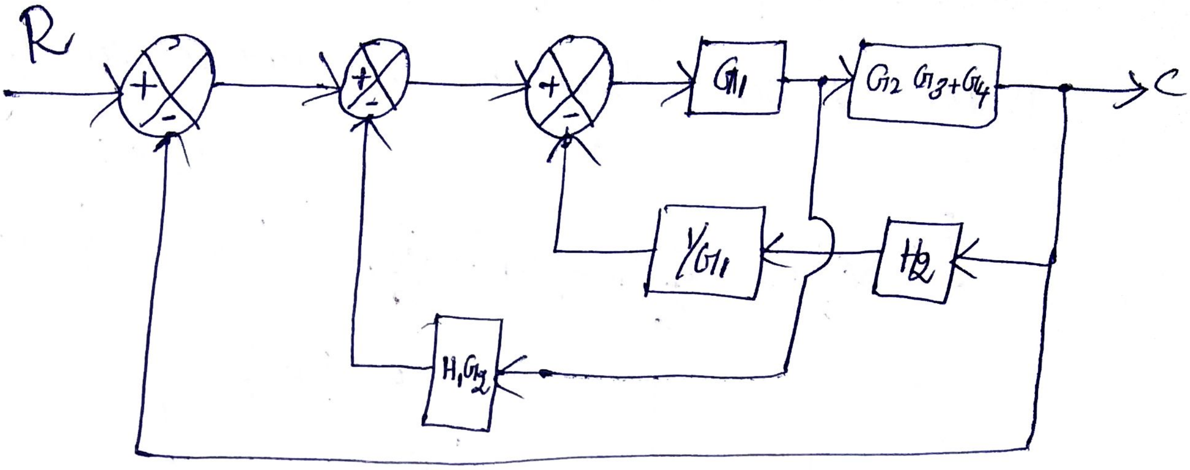
Step 1 :- Moving Branch point before the block



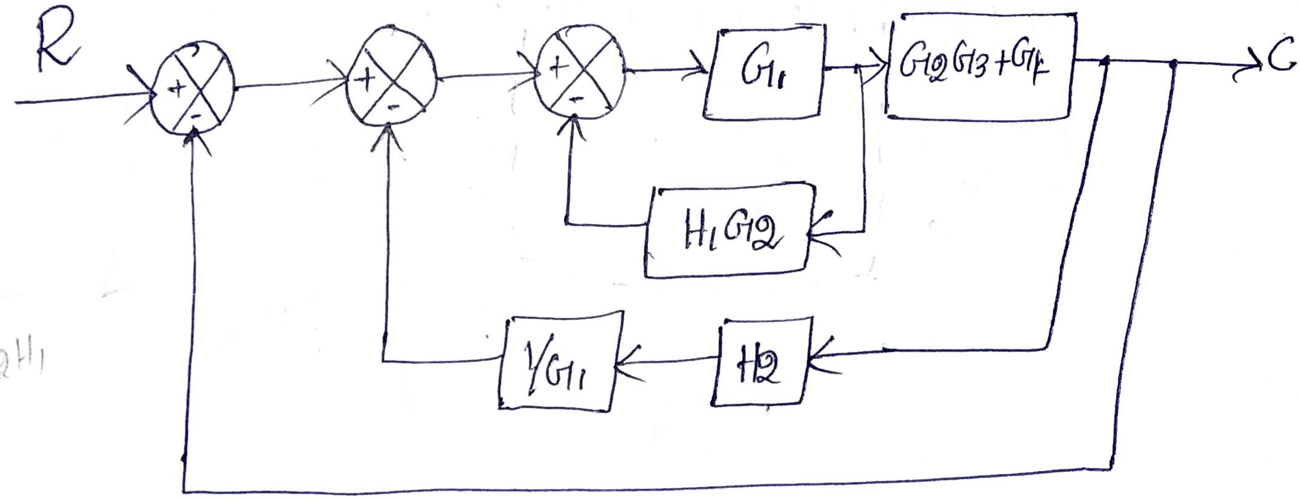
Step 2 :- Combining the Blocks in cascade & eliminating parallel blocks.



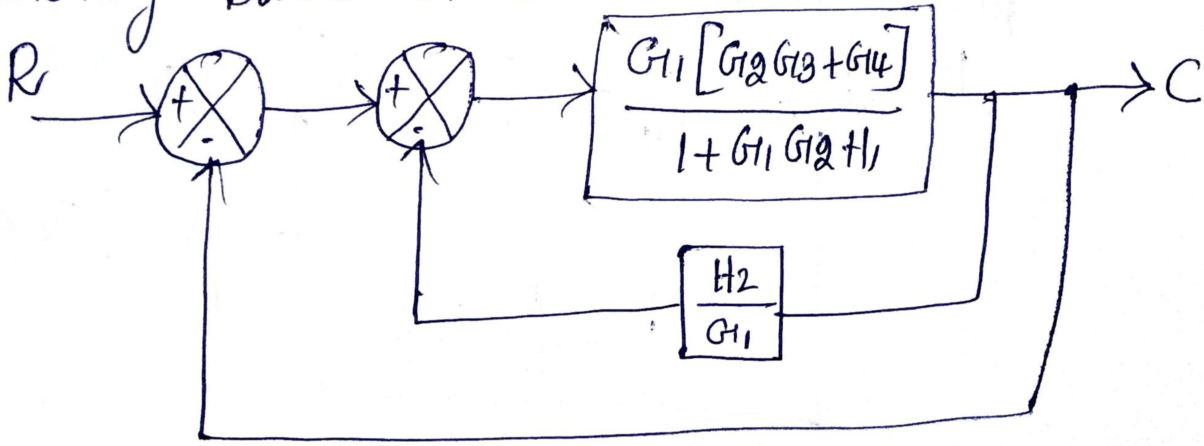
Step 3:- Moving Summing Point before the block



Step 4:- Interchanging summing points and Modifying Branch Points.



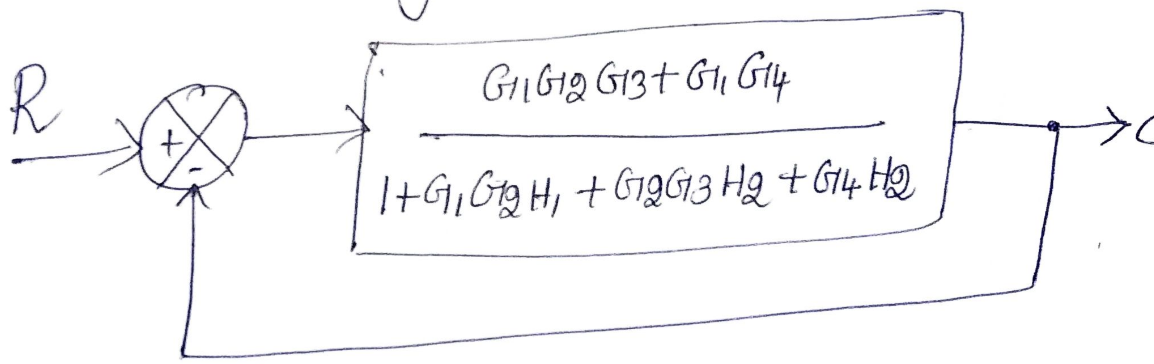
Step 5:- Eliminating the feedback path and Combining blocks in cascade



$$\frac{\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\dfrac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}\left(\dfrac{H_2}{G_1}\right)} \Rightarrow \frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1}}{\dfrac{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}{1+G_1G_2H_1}}$$

$$= \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}$$

Step 6: Eliminating the feedback path



$$\frac{C}{R} = \frac{\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}{1+\dfrac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2}}$$

$$\boxed{\frac{C}{R} = \frac{G_1G_2G_3+G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}}$$

Signal flow graph (SFG)

Signal flow graph approach and the block diagram approach yield the same information.

The advantage in signal flow graph method is that, using Mason's gain formula the overall gain of the system can be computed easily.

This method is simpler than the tedious block diagram reduction techniques.

Definition

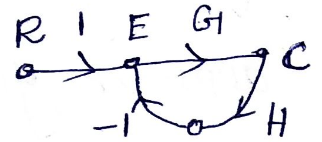
A SFG is a graphical representation of relationship between the variables of set of linear algebraic equations.

It consists of a network in which nodes representing each of system variables are connected by direct branches.

Some important terms in SFG are as follows.

Node:-

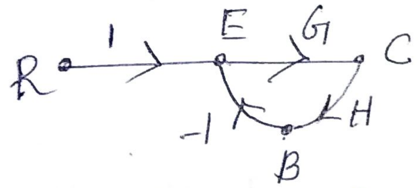
It represents a system variable which is equal to sum of all incoming signals.



Here R, E, C are nodes.

Branch:

A signal travels along a branch from one node to another in direction, indicated by the branch arrow.



Here G is branch.

Input node or Source

It is a node, only outgoing branches.

R → input node

Output Node or Sink

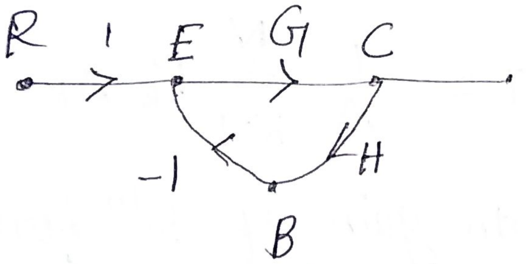
It is a node, with only incoming Branches. C → output node.

Path:-

It is the tra-versal of connected branches in the direction of branch arrows such that no node is tra-versed more than once.

Forward path:

It is the path from input node to Output node when no node encountered time.



R-E-C is a forward path

forward path gain

It is a product of branch gains in forward path.

G is a forward path gain

Loop:-

It is a path which originates and terminates at the same node.

Here ECBE is a loop

Loop gain

It is a product of branch gains encountered in traversing the loop

Here $(-GH)$ is the loop gain

Non touching loop

If they donot possess any common node. It is called non touching loop.

Mason's Gain formula

According to Mason's gain formula, the overall gain 'T' is expressed as

$$T = \frac{1}{\Delta} \sum_{K=1}^{N} P_K \Delta_K$$

Where

$P_K \rightarrow$ path gain of K^{th} forward path.

$\Delta \rightarrow$ determinant of the path

$\Delta = 1 -$ [Sum of loop gain of all individual loop]
+ [Sum of gain products of all possible combinations of two non touching loops]
− [Sum of gain products of all possible combinations of three non touching loops]
+ .....

$$\boxed{\Delta = 1 - \sum_{m} P_{m1} + \sum_{m} P_{m2} - \sum_{m} P_{m3} + \cdots}$$

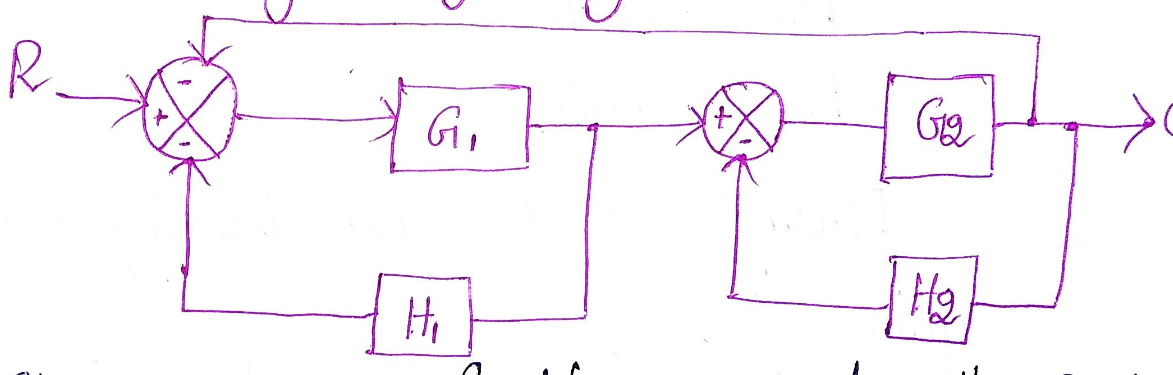
Δ_K = Value of Δ for that part of graph non touching the K^{th} forward path.

T = Overall gain of the system.

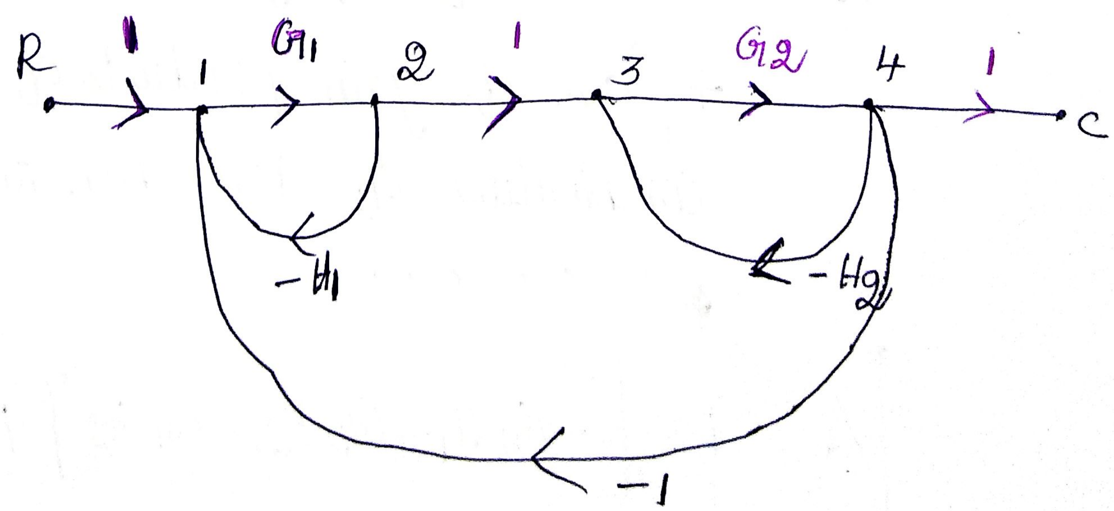
P_{mr} = gain product of m^{th} possible combination of r^{th} non touching loops

N = Total Number of forward paths.

1) Obtain the transfer function of a system given in block diagram by using signal flow graph.



Step 1:- The signal flow graph for the system is drawn below.



Step 2:

$P_K \rightarrow$ path gain K^{th} forward path

Here forward path is only one [R to C]

$\therefore \; P_1 = G_1 G_2$

Step 3

Finding out individual loops.

Here, there are 3 individual loops, gains are

$P_{11} = -G_1 H_1$

$P_{12} = -G_2 H_2$

$P_{13} = -G_1 G_2$

Finding out two non touching loops with gain

$P_{12} = (-G_1 H_1)(-G_2 H_2) = G_1 G_2 H_2 H_1$

$\Delta = 1 -$ [Sum of loop gain of all individual loops] + [Sum of gain products of all possible combination of two non touching loops]

$- \cdots$

$$\boxed{\Delta = 1 - [-G_1 H_1 - G_2 H_2 - G_1 G_2] + G_1 G_2 H_1 H_2}$$

Δ_K

There is one forward path and all loops touch the forward path

$$\boxed{\Delta_1 = 1}$$

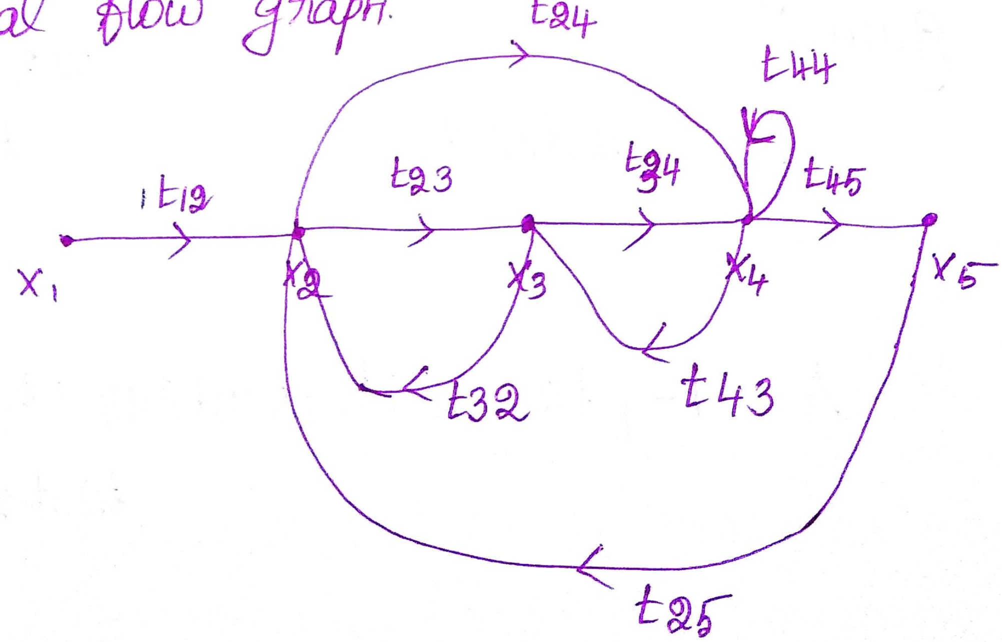
Mason's Gain formula

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$T = \frac{P_1 \Delta_1}{\Delta} = \frac{G_1 G_2 (1)}{1 + G_1 H_1 + G_2 H_2 + G_1 G_2 + G_1 G_2 H_1 H_2}$$

$$\boxed{T = \frac{G_1 G_2}{1 + G_1 H_1 + G_2 H_2 + G_1 G_2 + G_1 G_2 H_1 H_2}}$$

2) Determine the transfer function by using Signal flow graph.



Step 1:

$P_K \rightarrow$ Path gain K^{th} forward path

Here forward path is 3

$P_1 = t_{12}\, t_{23}\, t_{34}\, t_{45}$

$P_2 = t_{12}\, t_{24}\, t_{45}$

$P_3 = t_{12}\, t_{25}$

Step 2:

finding out individual loops

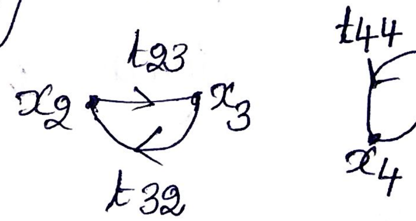
$P_{11} = t_{23}\, t_{32}$

$P_{12} = t_{34}\, t_{43}$

$P_{13} = t_{44}$

$P_{14} = t_{32}\, t_{43}\, t_{24}$

Finding Out two non touching loops with gain



$P_{12} = t_{23}\, t_{32}\, t_{44}$

$$\Delta = 1 - [t_{23}t_{32} + t_{34}\, t_{43} + t_{44} + t_{32}\, t_{43}\, t_{24}] + t_{23}t_{32}t_{44}$$

Step 3

Δ_K

There is 3 forward path. So finding out Δ_1, Δ_2 & Δ_3

The first forward path touches all loops, So no individual loop is formed $\boxed{\Delta_1 = 1}$

The third forward path is eliminated. no individual loop is formed

$$\boxed{\Delta_2 = 2}$$

The third forward path is eliminated, two loops are formed

$$\boxed{\Delta_3 = 1 - t_{34} t_{43} - t_{44}}$$

Mason's gain formula

$$T.F = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$= \frac{1}{\Delta} \left[P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 \right]$$

$$\boxed{T.F = \frac{(t_{12} t_{23} t_{34} t_{45}) + (t_{12} t_{24} t_{45}) + t_{12} t_{25} (1 - t_3 t_{43} - t_{44})}{1 - (t_{32} t_{23} + t_{34} t_{43} + t_{44} + t_{32} t_{43} t_{24}) + t_{32} t_{23} t_{44}}}$$

Servo Motor

The motors that are used in automatic Control System are called servomotors.

When the objective of the system is to Control the position of an object then the system is called servomechanism.

Servo Motor

- DC servoMotor
  - Sliding Contact Motor
    - permanent Magnet Motor
      1) Cylindrical Armature Motors
      2) Disc armature motors
      3) Moving coil motors
    - Electromagnetic field Motor
      1) Armature Control motor
      2) Field Controlled Motor
      3) Series Motor
      4) Split fied Motor.
  - Contact less Motor
- AC servomotor

Servo Motor

→ Convert electrical signal to Mechanical signal

→ Convert an electrical signal to angular velocity or Movement of Shaft.

Requirements of Good Servo Motor

1) It should be easily reversible, have linear torque characteristics.
2) Operation should be stable without any oscillation.
3) Linear relationship between Speed and electric Control signal.
4) Low Mechanical and electrical inertia.
5) Fast response.

DC Servo Motor

⇒ Same as DC Motor

⇒ Act as Mechanical transducer which conver DC Voltage into Mechanical signal.

⇒ Control of DC servo motor can be from

* Armature Side
* Field side

AC ServoMotor

Convert AC voltage into Mechanical signal.

Comparison Between DC & AC servo Motor

AC ServoMotor	DC ServoMotor
Operates with low power	Operates with high power
No brushes & gives Noiseless operations	Noisy operations
Low efficiency	High efficiency
More stable	Less stable
Less Maintenance required	More Maintenance Required

Synchros

The synchro is a type of transducer which transforms the angular position of the shaft into an electric signal.

It is most widely used for error detection in feedback control systems. It measures & compares two angular displacements and its output voltage is approximately linear with angular difference.

Multi Variable Control System

Control System

SISO MIMO

SISO – Single input Single Output

MIMO – Multiple input Multiple Output

Multivariable Control is a technique that allows us to deal with more than one Control objective at the same time.

Example: Automobile Driving System.

UNIT: II TIME RESPONSE ANALYSIS:

Transient & steady state response - Measures of Performance of first order & second order system - Effect of additional pole & additional zero - steady state error constant & system - Type number - PID controller - Analytical design of PI, PD, PID controller.

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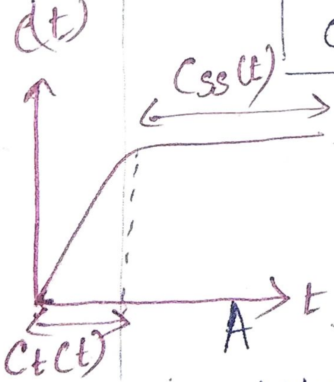
Introduction:

Time response analysis is also called time domain analysis. ie., output as a function of time.

Total time response $c(t)$ of a control system consists of transient response (dynamic response) $c_t(t)$ and steady state response $c_{ss}(t)$:



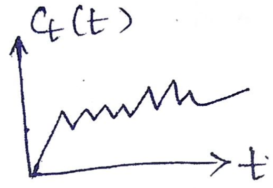
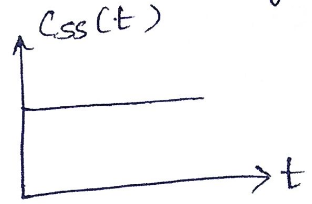
$$\boxed{c(t) = c_t(t) + c_{ss}(t)}$$

$c_{ss}(t)$ → steady state response

$c_t(t)$ → Transient response

$c(t)$ → Total time response.

A feedback control system has the inherent capabilities that its parameters can be adjusted to alter both its transient and steady-state behaviour.

Transient Response	Steady state Response
It remains for a very short time. 	It remains as time 't' approaches infinity (long time) 
It depends on system poles only. not on type of input.	It depends on both system poles and type of input.

Before proceeding with time response analysis of a control system, it is necessary to test stability of the system through indirect tests without actually obtaining the transient response. In case, system is unstable. hence of no practical use. we need not proceed with its transient response Analysis.

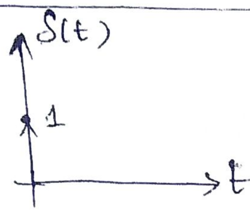
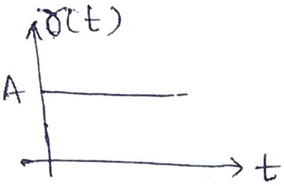
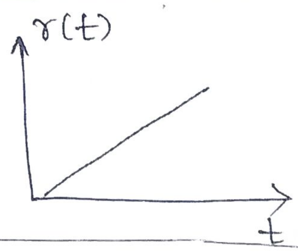
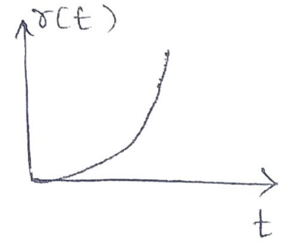
Typically test signals generated in laboratory are,

(i) unit Impulse (sudden shock)

(ii) unit step (sudden change)

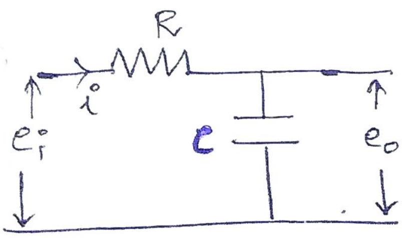
(iii) Ramp (constant velocity)

(iv) parabolic (constant Acceleration).

Sno	Signal	Diagram	Input $r(t)$	Output $R(S)$
1	unit Impulse		$r(t) = \delta(t) = \begin{cases} 1, & t=0 \\ 0, & t \neq 0 \end{cases}$	$R(S) = 1$
2.	unit step		$r(t) = u(t) = \begin{cases} A, & t>0 \\ 0, & t<0 \end{cases}$	$R(S) = \frac{1}{S}$
3.	Ramp		$r(t) = \begin{cases} t, & t>0 \\ 0, & t<0 \end{cases}$	$R(S) = \frac{1}{S^2}$
4.	parabolic signal		$r(t) = \begin{cases} \frac{t^2}{2}, & t>0 \\ 0, & o.w \end{cases}$	$R(S) = 1/S^3$

Measures of performance of First order System.

Let us consider a simple RC circuit,



RC circuit of first order s/m.

$$e_i = Ri + \frac{1}{C}\int i\,dt \longrightarrow (1)$$

$$e_o = \frac{1}{C}\int i\,dt \longrightarrow (2)$$

Taking Laplace transform for (1) & (2)

$$E_i(s) = \left[R + \frac{1}{Cs}\right] I(s) \longrightarrow (3)$$

$$E_o(s) = \frac{1}{Cs} I(s) \longrightarrow (4)$$

Transfer function: $\frac{E_o(s)}{E_i(s)} = \frac{(1/Cs)\,\cancel{I(s)}}{(R+1/Cs)\,\cancel{I(s)}}$

$$\frac{E_o(s)}{E_i(s)} = \frac{\cancel{Cs}}{(1+RCs)\,\cancel{Cs}} = \frac{1}{1+RCs} \longrightarrow (5)$$

where $\tau = RC$ = Time constant.

$$\therefore \boxed{T.F = \frac{1}{1+\tau s}}$$

$$\Rightarrow T.F = \frac{C(s)}{R(s)} = \frac{1}{1+\tau s}$$

$$\boxed{C(s) = R(s)\left(\frac{1}{1+\tau s}\right)} \longrightarrow \text{Main Equation.}$$

<u>Input : unit impulse s/g</u>

$$\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$$

$$R(s) = 1$$

$$\therefore C(s) = \frac{1}{1+\tau s} \longrightarrow (6) \qquad C(s) = \frac{1}{\tau(s+1/\tau)}$$

Taking inverse L.T $\Rightarrow$ $\boxed{c(t) = \frac{1}{\tau} e^{-t/\tau}} \longrightarrow (7)$

Steady state error:-

$$e_{ss} = \lim_{t\to\infty} [e(t)] = \lim_{t\to\infty} [r(t) - c(t)]$$

[or]

$$e_{ss} = \lim_{s\to 0} s[E(s)] = \lim_{s\to 0} s\,[R(s) - C(s)] \rightarrow 8$$

Substitute R(s) & C(s) value in 8.

$$\therefore e_{ss} = \lim_{s\to 0} s\left[1 - \frac{1}{\tau s+1}\right]$$

$$\boxed{e_{ss} = 0}$$

Input: unit step s/g

$R(s) = 1/s$ substitute in main eqn

$$\therefore C(s) = \left(\frac{1}{s}\right)\left(\frac{1}{1+\tau s}\right)$$

[Partial fraction Expansion]

$$\frac{1}{s(1+\tau s)} = \frac{A}{s} + \frac{B}{1+\tau s} \Rightarrow \frac{1}{s(1+\tau s)} = \frac{A(1+\tau s)+Bs}{s(1+\tau s)}$$

$1 = A(1+\tau s) + Bs \rightarrow 1$ put $s=0$

$\boxed{1 = A} \rightarrow 2$

put $s = -1/\tau$

$$1 = B\left(-\frac{1}{\tau}\right)$$

$\boxed{B = -\tau} \rightarrow 3$

$$\therefore \boxed{C(s) = \frac{1}{s} - \frac{\tau}{1+\tau s}}$$

Apply inverse L.T on above eqn.

$$C(s) = \frac{1}{s} - \frac{\tau}{1+\tau s}$$

$$C(s) = \frac{1}{s} - \frac{\tau}{\tau(s+1/\tau)}$$

$$\boxed{c(t) = 1 - e^{-t/\tau}}$$

Steady state error:-

$$e_{ss} = \lim_{s \to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s \to 0} s\left[\frac{1}{s} - \frac{1}{s(1+\tau s)}\right]$$

$$= \lim_{s \to 0} \left(1 - \frac{1}{1+\tau s}\right)$$

$$\boxed{e_{ss} = \cancel{\lim}\ 0 //.}$$

Input: unit Ramp signal.

$R(s) = 1/s^2$. $\Rightarrow$ substitute in main eqn.

$$\therefore C(s) = \frac{1}{s^2}\left(\frac{1}{1+\tau s}\right)$$

$$C(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{1+\tau s}.$$

$\cancel{1 = AS^2 + AS(\tau S+1) + CS^2}$

$1 = AS(\tau S+1) + B(1+\tau S) + CS^2$

$\boxed{\text{Put } S=0}$

$\boxed{1 = B}$

$\boxed{\text{Put } S = -1/\tau}$

$1 = C\left(-1/\tau\right)^2$

$\boxed{C = \tau^2}$

$\boxed{\text{Put } S=1}$

$1 = A(1+\tau) + B(1+\tau) + C$

$1 = (A+B)(1+\tau) + C$

$1 = (A+1)(1+\tau) + \tau^2$

$\cancel{1} = A\cdot(1+\tau) + \cancel{1} + \tau + \tau^2$

$-\tau(\cancel{1+\tau}) = A(\cancel{1+\tau})$

$\boxed{A = -\tau}$

$\therefore \quad C(S) = \frac{-\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{1+\tau S}$

$C(S) = \frac{-\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{\tau(S+1/\tau)}$

Take I.L.T,

$C(t) = -\tau + 1 + \tau e^{-t/\tau}$

$\boxed{C(t) = 1 - \tau(1 - e^{-t/\tau})}$

Steady state error :-

$$e_{ss} = \lim_{s \to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s \to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(\tau s+1)}\right]$$

$$= \lim_{s \to 0} \frac{1}{s}\left(1 - \frac{1}{\tau s+1}\right)$$

$$= \lim_{s \to 0} \frac{1}{\cancel{s}}\left(\frac{\tau \cancel{s}}{1+\tau s}\right)$$

$$\boxed{e_{ss} = \tau}$$

Response of first order system

Sno	Signal	input R(s)	output c(t)	Steady state error e_{ss}
1.	Impulse signal	1	$\frac{1}{\tau} e^{-t/\tau}$ Diff	0
2.	step signal	$1/s$	$1 - e^{-t/\tau}$ Integ.	0
3.	Ramp signal	$1/s^2$	$t - \tau(1 - e^{-t/\tau})$	τ

Differentiation of step signal ⇒ Impulse s/g.

$$\frac{d}{dt}(1-e^{-t/\tau}) \Rightarrow \frac{1}{\tau}e^{-t/\tau}.$$

Integration of step signal ⇒ Ramp signal.

$$c(t) = \int (1-e^{-t/\tau}) = t + \tau e^{-t/\tau} + C.$$

Assume initial conditions are zero,

$$c(0) = 0 + \tau + C.$$

$$\boxed{C = -\tau}$$

$$\therefore\ c(t) = \int (1-e^{-t/\tau}) = t + \tau e^{-t/\tau} - \tau \Rightarrow \text{Ramp s/g.}$$

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Measures of performance of Second order system.

The standard form of closed loop transfer function of second order system is given by,

$$T.F = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where $\omega_n \rightarrow$ Natural undamped freq (rad/sec)

$\xi \rightarrow$ damping ratio $= \dfrac{\text{Actual damping}}{\text{critical damping}}$ (no units)

Depend on Damping ratio 'ξ', system can be classified in to four cases:

(i) undamped system : $\xi = 0$

(ii) under damped system : $0 < \xi < 1$

(iii) critically damped system : $\xi = 1$

(iv) over damped system : $\xi > 1$

The characteristic equation of second order system,

$$\boxed{s^2 + 2\xi\omega_n s + \omega_n^2 = 0}$$

It is a Quadratic equation, & root of this eqn,

ax^2+bx+c Quadratic formula } $= \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a = 1, \; b = 2\xi\omega_n, \; c = \omega_n^2$

$\therefore$ roots $s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = \frac{-2\xi\omega_n \pm \sqrt{4\omega_n^2[\xi^2-1]}}{2}$

$= \frac{-2\xi\omega_n \pm 2\omega_n\sqrt{\xi^2-1}}{2}$

$$\boxed{s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}}$$ $\Rightarrow$ roots of second order eqn.

case(i) undamped ($\xi = 0$)

$\boxed{s_1, s_2 = \pm j\omega_n}$ (pure imaginary) $j = \sqrt{-1}$, $j^2 = -1$

Case (ii) under damped $0 < \xi < 1$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$$

$$\boxed{s_1, s_2 = -\xi\omega_n \pm j\,\omega_n\sqrt{1-\xi^2}}$$ (complex conjugate)

Case (iii) critically damped $\xi = 1$

$$\boxed{s_1, s_2 = -\omega_n}$$ (real & equal).

Case iv Over damped $\xi > 1$

$$\boxed{s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}}$$ (real & unequal)

In general, roots of second order s/m $\left.\right\}$ $\boxed{s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}}$

where $\sigma = \xi\omega_n \Rightarrow$ attenuation &

$\omega_n\sqrt{1-\xi^2} = \omega_d \Rightarrow$ frequency of damped oscillation (rad/sec).

$\therefore$ roots of Second order System $\left.\right\}$ $\boxed{s_1, s_2 = -\sigma \pm \omega_d}$ //.

Time Domain response of Second order system for unit Impulse signal.

Second order Transfer fn

$$\boxed{T = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$$

Input : impulse s/g. $\therefore$ $C(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

↳ Main eqn.

Case (i): undamped s/m ($\xi = 0$) sub in main eqn,

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} = \omega_n \cdot \frac{\omega_n}{s^2 + \omega_n^2}$$

Taking Inverse L.T,

$$\boxed{c(t) = \omega_n \sin \omega_n t}$$

Case (ii): critically damped s/m ($\xi = 1$) sub in main eqn.

$$C(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

Taking I.LT

$$\boxed{c(t) = \omega_n^2 \, t \, e^{-\omega_n t}}$$

case (iii): Overdamped system ($\xi > 1$) sub in main eqn.

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Roots $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$.

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$$

$$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

$$\therefore C(s) = \frac{A}{s - s_1} + \frac{B}{s - s_2} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = A(s - s_2) + B(s - s_1) \qquad \text{sub } \boxed{s = s_1}$$

$$\omega_n^2 = A(s_1 - s_2)$$

$$\boxed{A = \frac{\omega_n^2}{s_1 - s_2}} \quad \text{substitute } s_1 \text{ \& } s_2 \text{ values.}$$

$$A = \frac{\omega_n^2}{\cancel{-\xi\omega_n} + \omega_n\sqrt{\xi^2 - 1} + \cancel{\xi\omega_n} + \omega_n\sqrt{\xi^2 - 1}}$$

$$= \frac{\omega_n^{\cancel{2}}}{2\cancel{\omega_n}\sqrt{\xi^2 - 1}}$$

$$\boxed{A = \frac{\omega_n}{2\sqrt{\xi^2 - 1}}}$$

substitute $s = s_2$,

$$B = \frac{\omega_n^2}{s_2 - s_1}$$

similarly as like A,

$$\boxed{B = \frac{-\omega_n}{2\sqrt{\xi^2-1}}}$$

$$\therefore C(s) = \frac{\omega_n/2\sqrt{\xi^2-1}}{s-s_1} - \frac{\omega_n/2\xi^2-1}{s-s_2}$$

$$C(s) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

Take inverse Laplace Transform,

$$\boxed{C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(e^{s_1 t} - e^{s_2 t}\right)}$$

Substitute s_1 & s_2,

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t} - e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t}\right]$$

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[e^{-\xi\omega_n t}\left(e^{(\omega_n\sqrt{\xi^2-1})t} - e^{(-\omega_n\sqrt{\xi^2-1})t}\right]\right.$$

$$\boxed{C(t) = \frac{\omega_n}{\sqrt{\xi^2-1}} e^{-\xi\omega_n t}\left[\sinh \omega_n\sqrt{\xi^2-1}\right]t}$$

$$\left(\because \frac{e^{\theta}-e^{-\theta}}{2} = \sinh\theta\right)$$

case(iv): underdamped system $(0 < \xi < 1)$

Roots: $S_1, S_2 \Rightarrow -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$$S_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$S_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}.$$

$$C(S) = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2} = \frac{\omega_n^2}{(S-S_1)(S-S_2)}$$

$$\frac{\omega_n^2}{(S-S_1)(S-S_2)} = \frac{A}{S-S_1} + \frac{B}{S-S_2}.$$

$$\omega_n^2 = A(S-S_2) + B(S-S_1) \qquad \text{if } (S = S_1)$$

$$\omega_n^2 = A(S_1-S_2)$$

$$A = \frac{\omega_n^2}{S_1-S_2} = \frac{\omega_n^{\cancel{2}}}{2j\cancel{\omega_n}\sqrt{1-\xi^2}} = \boxed{\frac{\omega_n}{2j\sqrt{1-\xi^2}} = A}$$

IIIly $\boxed{B = \frac{-\omega_n}{2j\sqrt{1-\xi^2}}}$.

$$\therefore C(S) = \frac{\omega_n/2j\sqrt{1-\xi^2}}{S-S_1} - \frac{\omega_n/2j\sqrt{1-\xi^2}}{S-S_2}.$$

$$C(s) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

Take I.L.T

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left(e^{s_1 t} - e^{+s_2 t}\right)$$

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left(e^{(-\xi\omega_n + j\omega_n\sqrt{1-\xi^2})t} - e^{(-\xi\omega_n - j\omega_n\sqrt{1-\xi^2})t}\right)$$

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\cdot\left[e^{(j\omega_n\sqrt{1-\xi^2})t} - e^{(-j\omega_n\sqrt{1-\xi^2})t}\right]$$

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\left[\omega_n\sqrt{1-\xi^2}\right]t \qquad \left(\because \frac{e^{j\theta}-e^{-j\theta}}{2j} = \sin\theta.\right)$$

$$\boxed{c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin(\omega_n\sqrt{1-\xi^2})t}$$

<u>Steady state error</u>:- $e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right]$

$$e_{ss} = \lim_{s\to 0} s\left[1 - \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right]$$

$$e_{ss} = \lim_{s\to 0} s\left[1 - \frac{\omega_n^2}{\omega_n^2}\right] = 0$$

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Time Response of second order system for unit step signal.

T.F of a second order system } $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

unit step signal $\boxed{R(s) = 1/s}$

$\therefore C(s) = \frac{1}{s}\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right)$

Case (i) undamped, $\xi = 0$.

$C(s) = \frac{1}{s}\left(\frac{\omega_n^2}{s^2+\omega_n^2}\right) = \frac{A}{s} + \frac{Bs+C}{s^2+\omega_n^2} = \frac{A(s^2+\omega_n^2)+(Bs+C)s}{s(s^2+\omega_n^2)}$

$\omega_n^2 = A(s^2+\omega_n^2)+(Bs+C)(s)$

$s=0$

$\omega_n^2 = A\omega_n^2 \Rightarrow \boxed{A=1}$

Expand the eqn,

$\omega_n^2 = As^2 + A\omega_n^2 + Bs^2 + Cs$

Equating s^2 terms $\Rightarrow$ $0 = A+B$ $\therefore \boxed{B=-1}$

Equating s term $\Rightarrow \boxed{0 = C}$

$\therefore C(s) = \frac{1}{s} + \left(\frac{-s}{s^2+\omega_n^2}\right) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$

$L\{1\} = \frac{1}{s}$

$L\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$

Taking Inverse L.T,

$$\boxed{c(t) = 1 - \cos\omega_n t}$$

Case ii critically damped system ($\xi = 1$)

$$C(s) = R(s)\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right)$$

$$C(s) = \left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}\right) = \left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)$$

Apply partial fraction.

$$C(s) = \frac{1}{s}\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right) = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2 + Bs(s+\omega_n) + Cs.$$

$$\omega_n^2 = A\omega_n^2 \quad \therefore \boxed{A=1} \qquad \text{Put } (s=0)$$

Expand the equation.

$$\omega_n^2 = As^2 + A\omega_n^2 + 2As\omega_n + Bs^2 + Bs\omega_n + Cs.$$

Equating s^2 term

$$0 = A + B \quad \therefore \boxed{B=-1}$$

Equating s term.

$$0 = 2A\omega_n + B\omega_n + C.$$

$$0 = 2\omega_n - \omega_n + C$$

$$\boxed{C = -\omega_n}$$

$$\boxed{A = 1;\ B = -1;\ C = -\omega_n}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$L(te^{-at}) = \frac{1}{(s+a)^2}$$

Take inverse Laplace transform.

$$\boxed{c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}}$$

<u>Case (iii)</u> Overdamped system $(\xi > 1)$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

<u>roots</u> $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$.

$$\therefore C(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)}$$

Taking partial fraction.

$$C(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{A}{s} + \frac{B}{s-s_1} + \frac{C}{s-s_2} \rightarrow (1)$$

$$\omega_n^2 = A(s-s_1)(s-s_2) + Bs(s-s_2) + Cs(s-s_1).$$

$$\omega_n^2 = A(-s_1)(-s_2) \quad \therefore \boxed{A = \frac{\omega_n^2}{s_1 s_2}}$$ put $s=0$

Substitute s_1, s_2 values in (A) eqn

$$A = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(-\xi\omega_n - \omega_n\sqrt{\xi^2-1})}$$

$$A = \frac{\omega_n^2}{\xi^2\omega_n^2 - \omega_n^2(\xi^2-1)} \qquad \left(\because (a+b)(a-b) = a^2-b^2\right)$$

$$A = \frac{\cancel{\omega_n^2}}{\xi^2\cancel{\omega_n^2} - \xi^2\cancel{\omega_n^2} + \cancel{\omega_n^2}} = 1. \quad \therefore \boxed{A=1}$$

Put $s = s_1$, in partial fraction equation.

$$\omega_n^2 = B\, s_1(s_1 - s_2)$$

$$\boxed{B = \frac{\omega_n^2}{s_1(s_1-s_2)}}$$

substitute s_1, s_2 values in (B) eqn.

$$B = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(-\cancel{\xi\omega_n} + \omega_n\sqrt{\xi^2-1} + \cancel{\xi\omega_n} + \omega_n\sqrt{\xi^2-1})}$$

$$B = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(2\omega_n\sqrt{\xi^2-1})} = \frac{\omega_n^2}{\omega_n[-\xi+\sqrt{\xi^2-1}](2\omega_n\sqrt{\xi^2-1})}$$

$$B = \frac{1}{(-\xi + \sqrt{\xi^2-1})(2\sqrt{\xi^2-1})} \qquad \therefore \boxed{B = \frac{1}{2\sqrt{\xi^2-1}\,(-\xi+\sqrt{\xi^2-1})}}$$

Substitute $s = s_2$ in partial fraction eqn.

$$\omega_n^2 = C\, s_2(s_2 - s_1)$$

$$\boxed{C = \frac{\omega_n^2}{s_2(s_2-s_1)}}$$

substitute s_1, s_2 values in (C) eqn.

$$C = \frac{\omega_n^2}{-\xi_p\omega_n - \omega_n\sqrt{\xi_p^2-1}\left[-2\omega_n\sqrt{\xi_p^2-1}\right]}$$

$$\boxed{C = \frac{1}{2\sqrt{\xi_p^2-1}\,(\xi_p+\sqrt{\xi_p^2-1})}}$$

substitute A, B, C values in eq (1)

$$C(s) = \frac{1}{s} + \frac{1}{2\sqrt{\xi_p^2-1}\,(-\xi_p+\sqrt{\xi_p^2-1})}\left(\frac{1}{s-s_1}\right) + \frac{1}{2\sqrt{\xi_p^2-1}\,(\xi_p+\sqrt{\xi_p^2-1})}\left(\frac{1}{s-s_2}\right)$$

Taking inverse laplace transform.

$$\boxed{c(t) = 1 + \frac{1}{2\sqrt{\xi_p^2-1}\,(-\xi_p+\sqrt{\xi_p^2-1})}e^{s_1 t} + \frac{1}{2\sqrt{\xi_p^2-1}\,(\xi_p+\sqrt{\xi_p^2-1})}e^{s_2 t}}$$ //.

<u>case (iv)</u>: underdamped system ($\xi_p < 1$)

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi_p\omega_n s+\omega_n^2)}$$

Apply partial fraction,

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi_p\omega_n s+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi_p\omega_n s+\omega_n^2}$$

$$\omega_n^2 = A(s^2+2\xi_p\omega_n s+\omega_n^2) + (Bs+C)(s)$$

$s=0$

$$\omega_n^2 = A\,\omega_n^2 \Rightarrow \boxed{A=1}$$

Expand the partial fraction eqn.

$$\omega_n^2 = AS^2 + 2A\xi\omega_n s + A\omega_n^2 + BS^2 + CS.$$

Equating s^2 terms.

$$0 = A + B \quad \therefore \boxed{B = -1}$$

Equating s terms.

$$0 = 2A\xi\omega_n + C$$

$$0 = 2\xi\omega_n + C \quad \therefore \boxed{C = -2\xi\omega_n}$$

substitute A, B, C values.

$$C(s) = \frac{1}{s} + \frac{(-s + (-2\xi\omega_n))}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Add & subtract $\xi^2\omega_n^2$ in denominator of second term.

$$C(s) = \frac{1}{s} - \left(\frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2 + \xi^2\omega_n^2 - \xi^2\omega_n^2}\right)$$

$$= \frac{1}{s} - \left(\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)}\right)$$

Let $\omega_d = \omega_n\sqrt{1 - \xi^2}$

$$\therefore \boxed{\omega_d^2 = \omega_n^2(1 - \xi^2)}$$

$$\therefore C(s) = \frac{1}{s} - \left(\frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}\right)$$

$$L\left[e^{-at}\sin\omega t\right] = \frac{\omega}{(s+a)^2+\omega^2}$$

12

$$L\left[e^{-at}\cos\omega t\right] = \frac{s+a}{(s+a)^2+\omega^2}$$

$$C(s) = \frac{1}{s} - \left(\frac{s+\xi\omega_n+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}\right)$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n(\omega_d)}{(\omega_d)\left((s+\xi\omega_n)^2+\omega_d^2\right)}$$

Taking Inverse Laplace Transform.

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t.$$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right].$$

substitute $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos(\omega_n\sqrt{1-\xi^2})t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_n\sqrt{1-\xi^2}t\right]$$

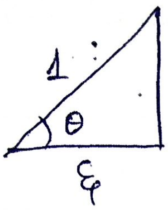
$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[(\sqrt{1-\xi^2})\cos\omega_n\sqrt{1-\xi^2}t + \xi\sin(\omega_n\sqrt{1-\xi^2})t\right]$$

Rearrange the terms,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right)$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\xi\sin\omega_d t + \sqrt{1-\xi^2}\cos\omega_d t\right).$$

On constructing right angle triangle with ξ_p and $\sqrt{1-\xi_p^2}$ we get,



$$\therefore \sin\theta = \sqrt{1-\xi_p^2}$$

$$\cos\theta = \xi_p$$

$$\tan\theta = \frac{\sqrt{1-\xi_p^2}}{\xi_p} \Rightarrow \theta = \tan^{-1}\frac{\sqrt{1-\xi_p^2}}{\xi_p}$$

Let us express $c(t)$ in standard form,

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi_p^2}}\left[(\sin\omega_d t \times \cos\theta) + (\cos\omega_d t \times \sin\theta)\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi_p^2}}\sin(\omega_d t + \theta)$$

$\therefore \boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$

where $\theta = \tan^{-1}\frac{\sqrt{1-\xi_p^2}}{\xi_p}$

$$\therefore \boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi_p^2}}\sin(\omega_d t + \theta)}$$

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Steady state error of second order s/m with unit step

$$e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s\to 0} s\left[\frac{1}{s} - \frac{\omega_n^2}{s(s^2+2\xi_p\omega_n s+\omega_n^2)}\right]$$

$$= \lim_{s\to 0}\left[1 - \frac{\omega_n^2}{\omega_n^2}\right] = 0$$

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SUMMARY

SECOND ORDER SYSTEM WITH IMPULSE INPUT

Sno	Damping Type	c(t)
1.	undamped ($\xi=0$)	$c(t) = \omega_n \sin\omega_n t$
2.	critically damp ($\xi=1$)	$c(t) = \omega_n^2 \, t \, e^{-\omega_n t}$
3.	overdamped ($\xi>1$)	$c(t) = \frac{\omega_n}{\sqrt{\xi^2-1}} e^{-\xi\omega_n t} \left[\sinh \omega_n \sqrt{\xi^2-1}\right] t$
4.	underdamped ($0<\xi<1$)	$c(t) = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_n \sqrt{1-\xi^2})t$

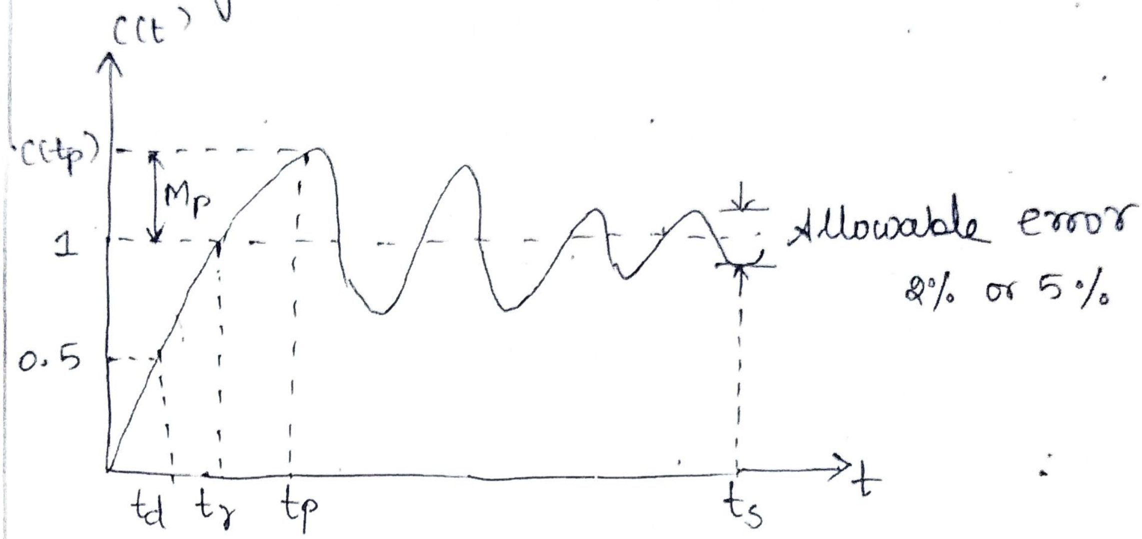
SECOND ORDER SYSTEM WITH UNIT STEP INPUT

Sno	Damping Type	c(t)
1.	undamped ($\xi=0$)	$c(t) = 1 - \cos\omega_n t$
2.	critically damp ($\xi=1$)	$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$
3.	overdamped ($\xi>1$)	$c(t) = 1 + \frac{1}{2\sqrt{\xi^2-1}(-\xi+\sqrt{\xi^2-1})} e^{S_1 t} + \frac{1}{2\sqrt{\xi^2-1}(\xi+\sqrt{\xi^2-1})} e^{S_2 t}$
4.	underdamped ($0<\xi<1$)	$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t+\theta)$

In Both the inputs, steady state error $\boxed{e_{ss}=0}$ //.

Transient Response Specifications:

The transient response of a system to a unit step input depends on initial conditions. ∴ to compare the time response of various system, it is necessary to start with standard initial conditions.



where, $t_d \rightarrow$ delay time

$t_r \rightarrow$ Rise time.

$t_p \rightarrow$ Peak time

$M_p \rightarrow$ Maximum peak overshoot

$t_s \rightarrow$ Settling time.

The time domain specifications are defined as follows,

(i) Delay time: (t_d):

It is the time required by the response to reach half of its final value at the first attempt.

(ii) Rise time :(t_r):

It is the time required for response to rise from 10% to 90% for overdamped or critically damped s/m and 0% to 100% for underdamped of its final value. ~~the~~ 5% to 95% of its ~~its~~ final value for may be used for critically damped system.

In general, unit step response of second order for under damped is given by,

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

Rise time $\Rightarrow \boxed{t = t_r}$ $\boxed{C(t_r) = 1}$

$$C(t_r) = \cancel{1} - \frac{e^{-\xi\omega_n t_r}}{\sqrt{1-\xi^2}} \sin(\omega_d t_r + \theta) = \cancel{1}$$

The term $\sin(\omega_d t_r + \theta) = 0 \Rightarrow \sin\phi = 0$.

when $\phi = 0, \pi, 2\pi \ldots$

$$\therefore \omega_d t_r + \theta = \pi$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}} = \frac{\pi - \theta}{\omega_n\sqrt{1-\xi^2}}$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$ unit is degree.

but 'π' unit is radian/~~sec~~. $\quad rad = \left(deg \cdot \frac{\pi}{180°}\right)$

(iii) Peak time : (t_p) :-

Peak time is obtained by differentiating c(t) with respect to t and equating to zero. At maxima, the slope is zero.

$$\therefore t_p = \left.\frac{dc(t)}{dt}\right|_{t=t_p} = 0$$

(After differentiate c(t))

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

"The time required for the response to reach first peak of overshoot".

where $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$

(iv) Maximum peak overshoot : (M_p)

It is maximum peak value of the response measured from unity.

$$\therefore M_p = c(t_p) - 1 = e^{-\xi\pi/\sqrt{1-\xi^2}}$$

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100$$

$c(t_p) \rightarrow$ peak value of c(t)

$c(\infty) \rightarrow$ final value of c(t).

$$\therefore \boxed{M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}}$$

(v) Settling time : (t_s):

It is the time required for response curve to reach and stay within a specified tolerance band (either 2% or 5%) of final value.

Settling time t_s ~~for 2% tolerance band~~

$t_s = \frac{4}{\xi\omega_n}$ for 2% tolerance band & $\boxed{\xi = 0.76}$

$t_s = \frac{3}{\xi\omega_n}$ for 5% tolerance band & $\boxed{\xi = 0.68}$

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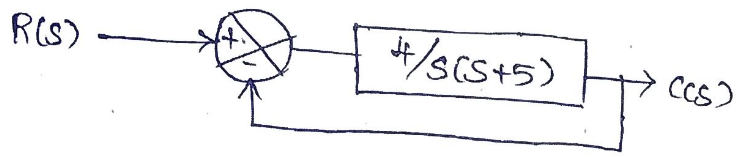
Problems on Response of the system

1. Obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when input is unit step.

⇒ solution :-



(Always prefer negative feedback)

For unity feedback,

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{4/s(s+5)}{1 + 4/s(s+5)}$$

$$\frac{C(s)}{R(s)} = \frac{4}{s^2+5s+4} = \frac{\cancel{4}}{\cancel{s}+4}$$

$$C(s) = R(s)\left(\frac{4}{s^2+5s+4}\right) = \frac{1}{s}\left(\frac{4}{s^2+5s+4}\right)$$

$$C(s) = \frac{4}{s(s+4)(s+1)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+1}.$$

$$4 = A(s+4)(s+1) + B(s)(s+1) + C(s+4)s.$$

Put $s=0 \Rightarrow \boxed{A=1}$

$s=-4 \Rightarrow \boxed{B=1/3}$

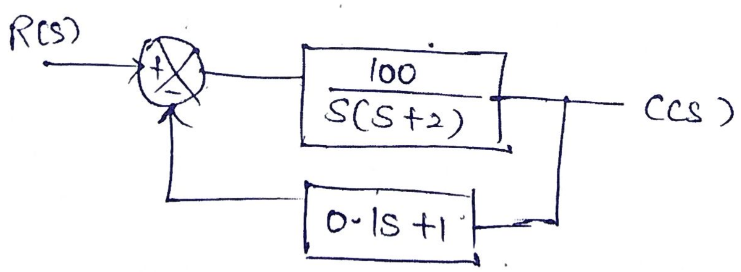
$s=-1 \Rightarrow \boxed{C=-4/3}$

$$\therefore C(s) = \frac{1}{s} + \frac{1}{3(s+4)} - \frac{4}{3(s+1)}$$

Take inverse Laplace transform,

$$\boxed{c(t) = 1 + \frac{1}{3}e^{-4t} - \frac{4}{3}e^{-t}}.$$

2. A position control system with velocity feedback is shown in figure. Calculate rise time, peak time, peak overshoot, settling time and also sketch the response.



= solution:-

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{100/s(s+2)}{1+\left(100/s(s+2)\right)(0.1s+1)}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{100}{s^2+12s+100}} \longrightarrow ①$$

General formula for 2nd order system,

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}} \longrightarrow ②$$

compare ① & ②, $\therefore \omega_n^2 = 100 \Rightarrow \boxed{\omega_n = 10\,rad/sec}$

$2\xi\omega_n = 12 \Rightarrow \boxed{\xi = 0.6}$ no unit

(i) rise time :-

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

$$\theta = \tan^{-1}\left(\sqrt{1-\xi^2}/\xi\right)$$

$$\theta = \tan^{-1}\left(\sqrt{1-0.6^2}/0.6\right) = \tan^{-1}(1.333) = 53.13^\circ$$

$$\theta = 53.13 \times \frac{\pi}{180} = 0.92 \text{ radian}.$$

$\omega_d = \omega_n\sqrt{1-\xi^2} = 8\ rad/sec$ $\quad \therefore t_r = \frac{\pi-0.92}{8} = 0.27\,sec$

(ii) peak time : $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{8} = 0.39$ sec.

(iii) Max. peak over shoot : $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$$= e^{-0.6\pi/\sqrt{1-0.6^2}}$$

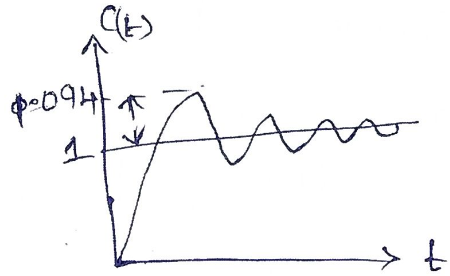
$$\boxed{\begin{array}{c} M_p = 0.094 \\ \% M_p = 9.4\% \end{array}}$$

(iv) settling time (t_s) :-

$2\%\ t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.6\times10} = 0.66$ sec.

$5\%\ t_s = \frac{3}{\xi\omega_n} = \frac{3}{0.6\times10} = 0.5$ sec.

(v) Response :-



3) The response of a servomechanism is $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step input. obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

= Solution:-

$$C(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}.$$

Take laplace transform,

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}.$$

$$C(s) = \frac{(s+60)(s+10) + 0.2s(s+10) - (1.2)s(s+60)}{s(s+60)(s+10)}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

Input : unit step signal, $R(s) = 1/s$.

$$C(s) = \frac{1}{s}\left(\frac{600}{s(s+60)(s+10)}\right)$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}}$$

In general, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

compare, $\omega_n^2 = 600 \Rightarrow \boxed{\omega_n = 24.49 \text{ rad/sec}}$

$$2\xi\omega_n = 70$$

$$\boxed{\xi = 1.42}$$

∴ undamped natural freq $= \omega_n = 24.49$ rad/sec.
Damping ratio $= \xi = 1.42$ //

4) The unity feedback system is characterized by an open loop transfer function $G(s) = K/s(s+10)$. Determine gain K, so that system will have damping ratio of 0.5 for this value of K. Determine settling time, peak overshoot and time at peak overshoot for a unit step input.

= Solution: $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$

$$= \frac{K/s(s+10)}{1+K/s(s+10)} = \frac{K}{s^2+10s+K}$$

by comparing with => $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

we get, $\omega_n^2 = K$; $2\xi\omega_n = 10$ $\quad \xi = \frac{10}{2\omega_n} = \frac{10}{2\sqrt{K}}$

$\boxed{\omega_n = \sqrt{K}}$ $\quad \xi = 0.5 \rightarrow$ given.

$\xi = \frac{10}{2\sqrt{K}} \Rightarrow 0.5 = \frac{5}{\sqrt{K}}$

$\boxed{K = 100}$ $\quad \therefore \boxed{\omega_n = 10 \text{ rad/sec}}$

Peak overshoot :- $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.163$

$\% M_p = 16.3\%$

Peak time :- $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.363$ sec //

5) The open loop transfer function of a unity feedback system is given by $G(s) = K/s(sT+1)$, where K & T are positive constant. By what factor should the amplifier gain k be reduced, so that peak overshoot of unit step response of the system is reduced from 75% to 25%.

⇒ Solution: $\frac{C(s)}{R(s)} = \frac{K/s(sT+1)}{1+K/s(sT+1)} = \frac{K}{Ts^2+s+K} = \frac{K/T}{s^2+1/T\,s+K/T}$

compare, $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi_p\omega_n s+\omega_n^2} = \frac{K/T}{s^2+1/T\,s+K/T}$

on comparing, we get,

$\omega_n^2 = K/T$; $2\xi_p\omega_n = 1/T$

$\omega_n = \sqrt{K/T}$ $\quad \xi_p = \frac{1}{2\sqrt{\frac{K}{T}}\,T} = \frac{1}{2\sqrt{KT}}$

$$\boxed{\omega_n = \sqrt{K/T}\ ;\ \xi_p = 1/2\sqrt{KT}}$$

Peak overshoot M_p is reduced (75% to 25%) by increasing ξ_p. The ξ_p is increased by reducing the k.

$$\boxed{M_p = e^{-\xi_p\pi/\sqrt{1-\xi_p^2}}}$$

Taking natural logarithm on both sides,

$$\ln M_p = -\xi_p\pi/\sqrt{1-\xi_p^2}$$

Squaring on both sides,

$$(\ln M_p)^2 = \xi_p^2 \pi^2 / 1-\xi_p^2$$

$$(\ln M_p)^2 (1-\xi_p^2) = \xi_p^2 \pi^2$$

$$(\ln M_p)^2 - \xi_p^2 (\ln M_p)^2 = \xi_p^2 \pi^2$$

$$(\ln M_p)^2 = \xi_p^2 (\pi^2 + (\ln M_p)^2)$$

$$\xi_p^2 = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

but $\xi_p = \frac{1}{2\sqrt{KT}} \Rightarrow \boxed{\xi_p^2 = \frac{1}{4KT}}$

$$\therefore \frac{1}{4KT} = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

$$\boxed{K = \frac{\pi^2 + (\ln M_p)^2}{4T(\ln M_p)^2}}$$

At $M_p = 0.75$, $K = K_1 = \frac{\pi^2 + (\ln 0.75)^2}{4\pi (\ln 0.75)^2}$

$$\boxed{K_1 = 30.1/T}$$

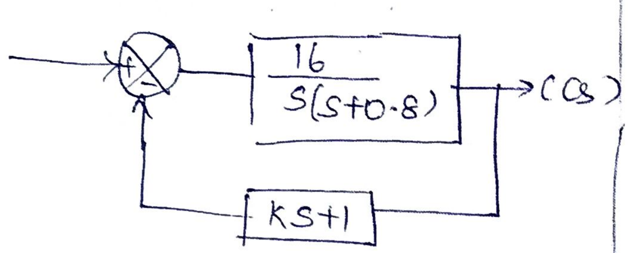
At $M_p = 0.25$, $K = K_2 = \frac{\pi^2 + (\ln 0.25)^2}{4\pi (\ln 0.25)^2}$

$$\boxed{K_2 = 1.53/T}$$

$\therefore \frac{K_1}{K_2} = 19.6 \Rightarrow K_2 = \frac{K_1}{19.6} \Rightarrow$ Gain k is reduced by 19.6 times the first gain//

6) A position control system with velocity feedback as shown. Given that $\xi = 0.5$. Also calculate rise time, Peak time, maximum overshoot and settling time.

: Solution :-



$$T.F = \frac{C(S)}{R(S)} = \frac{16/S(S+0.8)}{1+\left(16/S(S+0.8)\right)(KS+1)}$$

$$T.F = \frac{16}{S^2+(0.8+16K)S+16}$$

on comparing we get,

$\omega_n^2 = 16$ $\quad$ $0.8+16K = 2\xi\omega_n$.

$\boxed{\omega_n = 4 \text{ rad/sec}}$ $\quad$ $\xi = 0.5$ (given)

$\therefore \boxed{K = 0.2}$

$$T.F = \frac{C(S)}{R(S)} = \frac{16}{S^2+4S+16}$$

(i) Damped frequency : $\omega_d = \omega_n\sqrt{1-\xi^2} = 3.464$ rad/sec.

(ii) rise time : $t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-1.047}{3.464} = 0.6046$.

(Assume $\theta = 1.047$) $\Rightarrow \theta = \tan^{-1}\sqrt{1-\xi^2}/\xi$

(iii) peak time : $t_p = \frac{\pi}{\omega_d} = 0.907$ sec.

(iv) $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$= 0.163$

$\boxed{\% M_p = 16.3\%}$

(v) Settling time :-

For 2% error, $t_s = \dfrac{4}{\xi\omega_n} = 2\,sec$

For 5% error, $t_s = \dfrac{3}{\xi\omega_n} = 1.5\,sec$.

7) A unity feedback control system is characterized by following open loop transfer function $G(s) = (0.4s+1)/s(s+0.6)$. Determine transient response for unit step input.

= Solution :-

$$T.F = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{(0.4s+1)/s(s+0.6)}{1+((0.4s+1)/s(s+0.6))} = \frac{0.4s+1}{s^2+s+1}$$

$$C(s) = \left(\frac{1}{s}\right)\left(\frac{0.4s+1}{s^2+s+1}\right)$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$0.4s+1 = A(s^2+s+1) + (Bs+C)s$.

$\boxed{A=1}$ when $s=0$,

Equating s^2 terms, $0 = A+B \Rightarrow \boxed{B=-1}$

Equating s term, $0.4 = A + C$ $\therefore \boxed{C = -0.6}$

$$C(S) = \frac{1}{S} - \frac{S+0.6}{S^2+S+1} = \frac{1}{S} - \frac{S+0.6}{(S^2+2\times0.5S+0.5^2)+0.75}$$

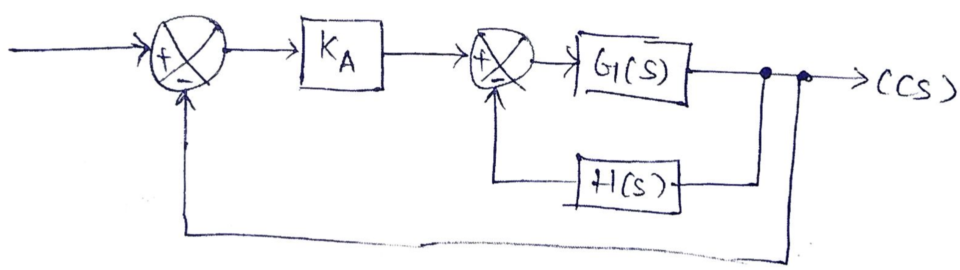
$$C(S) = \frac{1}{S} - \frac{S+0.5}{(S+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}}\frac{\sqrt{0.75}}{(S+0.5)^2+0.75}$$

Take inverse L.T,

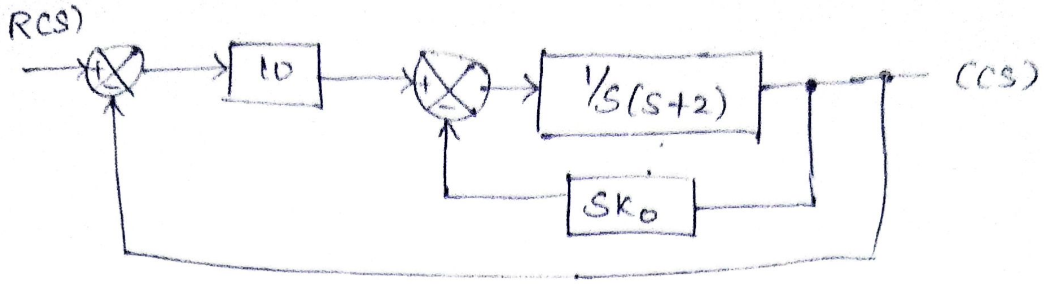
$$C(t) = 1 - e^{-0.5t}\cos\sqrt{0.75}\,t - \frac{0.1}{\sqrt{0.75}}e^{-0.5t}\sin\sqrt{0.75}\,t$$

8. A unity feedback control system has an amplifier with gain $K_A = 10$ and gain ratio, $G(S) = 1/S(S+2)$ in feedforward path. A derivative feedback $H(S) = SK_0$ is introduced as a minor loop around $G(S)$. Determine derivative feedback constant K_0, so that the system damping factor is 0.6.

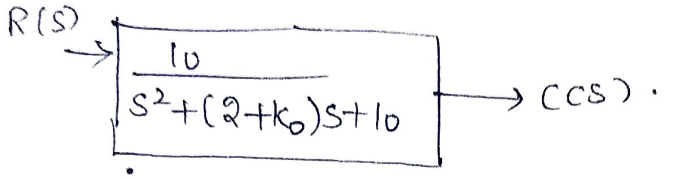
= solution:-



where $K_A = 10$, $G(S) = \frac{1}{S(S+2)}$, $H(S) = SK_0$.



⇓ After simplification



$$\therefore \quad \frac{C(S)}{R(S)} = \frac{10}{s^2+(2+k_0)s+10}$$

On compare, $\dfrac{C(S)}{R(S)} = \dfrac{\omega_n^2}{s^2+2\xi_p\omega_n s+\omega_n^2}$

$\omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10} = 3.162$ rad/sec.

$2 + k_0 = 2\xi_p\omega_n$.

$K_0 = 2 \times 0.6 \times 3.162 - 2 = 1.7944$.

$\boxed{K_0 = 1.7944}$

9) A closed loop servo is represented by the differential equation $\dfrac{d^2c}{dt^2} + 8\dfrac{dc}{dt} = 64e$. where c is displacement of output shaft, r is displacement of input shaft and $e = r-c$. Determine undamped natural frequency, damping ratio, M_p for unit step input of 12 units.

= Solution :-

The mathematical equation governing the System are,

$$\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64e.$$

Put $\boxed{e = r - c}$ $\Rightarrow$ $\frac{d^2c}{dt^2} + 8\frac{dc}{dt} = 64(r-c).$

Taking Laplace transform,

$$s^2C(s) + 8sC(s) = 64[R(s) - C(s)]$$

$$[s^2 + 8s + 64]C(s) = 64R(s)$$

$$\therefore \quad T.F = \frac{C(s)}{R(s)} = \frac{64}{s^2+8s+64}.$$

Now, compare with second order s/m.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$\therefore$ Natural undamped frequency : $\boxed{\omega_n = 8 \text{ rad/sec}}$

Damping ratio : $\boxed{\xi = 0.5}$

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.163.$$

and it input is 12 $\boxed{12 \text{ units}}$ $\therefore M_p = 0.163 \times 12 = 1.956$

$\% M_p = 1.956 \times 100 = 195.6\%$

— x — x —

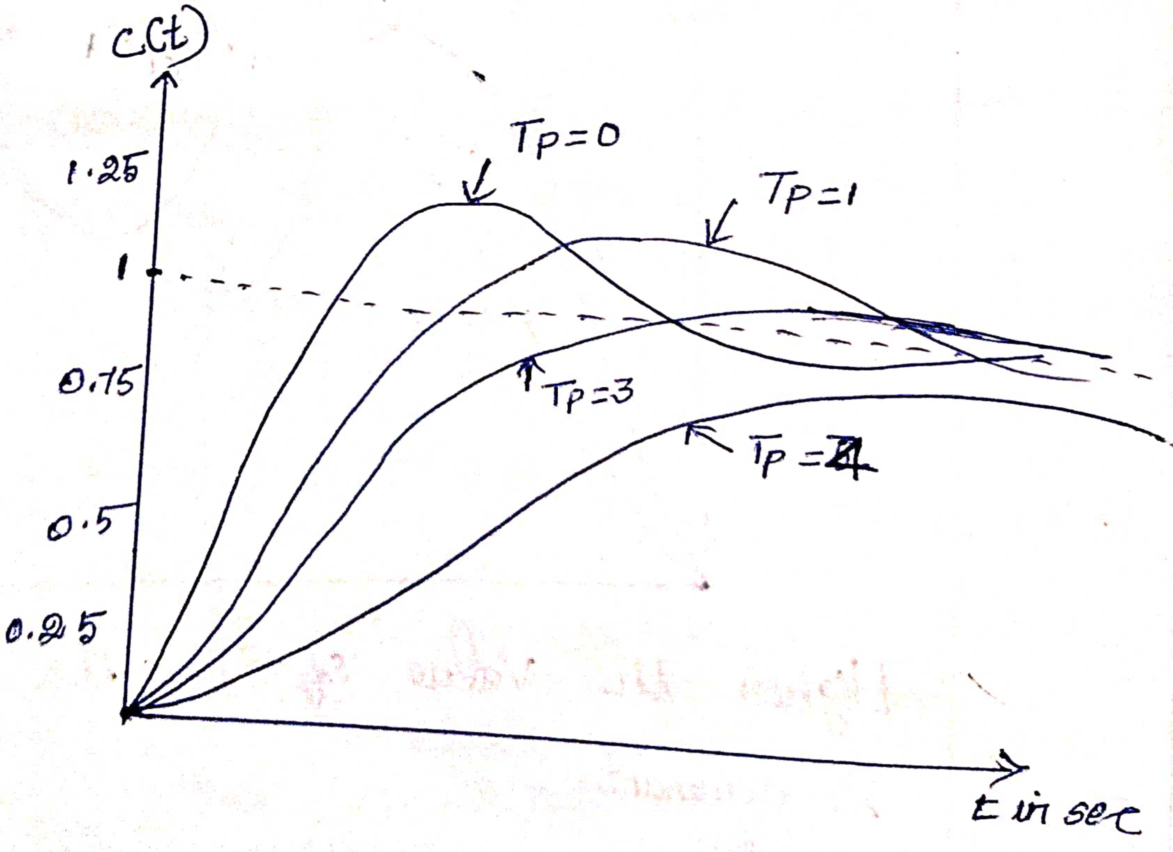
Effect of Addition of pole to closed loop transfer function

Consider a standard closed loop second order control system, with an added pole at as

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2 + 2\xi\omega_n s + \omega_n^2)(1 + T_p s)}$$

for $T_p = 0$. it behaves as standard second order system with an overshoot 16.3%.

for $T_p = 1$ onwards, the response become less oscillatory and the maximum overshoot decreases.



As T_p increases, the pole moves closer to the origin, which has following effects:

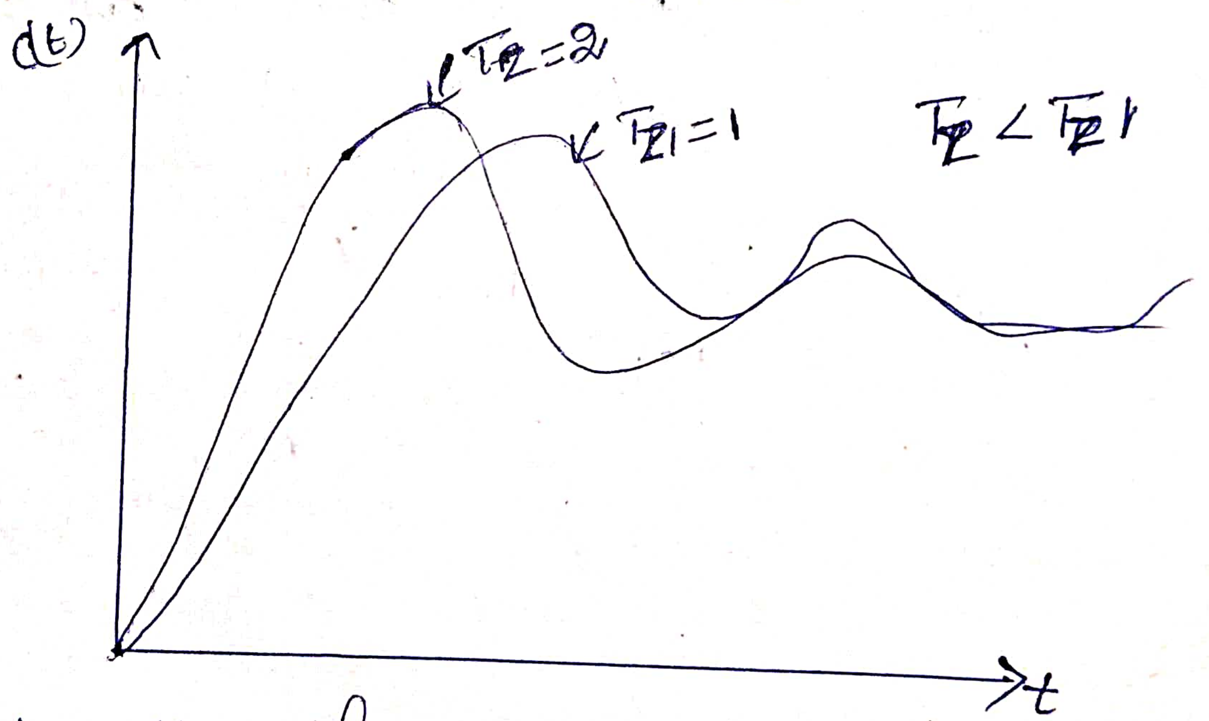
1) The maximum overshoot of the closed loop system decreases.

2) The rise time of the closed loop response increases.

Effect of adding zero to second order system

Consider a standard closed loop second order control system with an added zero as

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2(1+T_z s)}{s^2+2\xi\omega_n s+\omega_n^2}$$



Higher the value of T_z, the peaking phenomenon is dominant.

Introduction to P-I-D Controllers.

Proportional Controller (P)

In this Control mode, the output of the controller is simple proportional to the error e(t).

The relation between the error e(t) and the Controller output p is determined by Constant called proportional gain constant denoted as Kp.

$$P(t) = K_p e(t) + P_0$$

Integral Controller (I)

the Value of the Controlled output P(t) is changed at a rate which is proportional to the actuating error signal e(t). Mathematically it is expressed as

$$\frac{dp(t)}{dt} = K_i e(t)$$

$$P = K_i \int e(t)\, dt + P(0)$$

Derivative Controller (D)

In this mode, the output of the controller depends on the time rate of change of the actual errors. Hence it is also called rate action mode or anticipatory action mode

$$P(t) = K_d \cdot \frac{de(t)}{dt}$$

$K_d \rightarrow$ derivative gain constant.

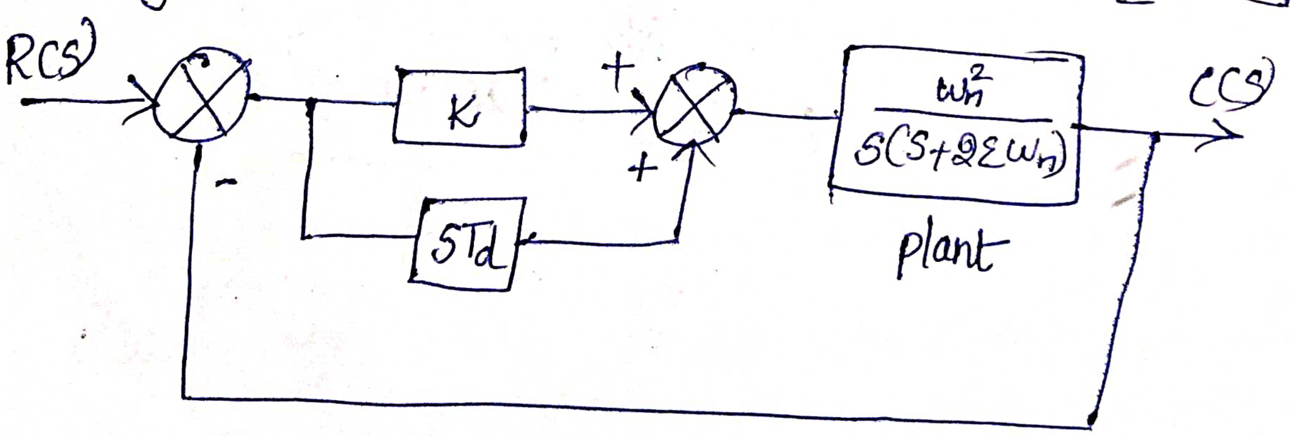
Analytical Design for PD, PI, PID Control Systems.

PD type of Controller.

A controller in the forward path, which changes the controller output corresponding to proportional plus derivative of error signal is called PD controller.

$$\text{Output of Controller} = Ke(t) + T_d \frac{de(t)}{dt}$$

$$\text{Taking Laplace} = K\,E(S) + S T_d\,E(S) = E(S)\left[K + S T_d\right]$$



PI Type of Controller.

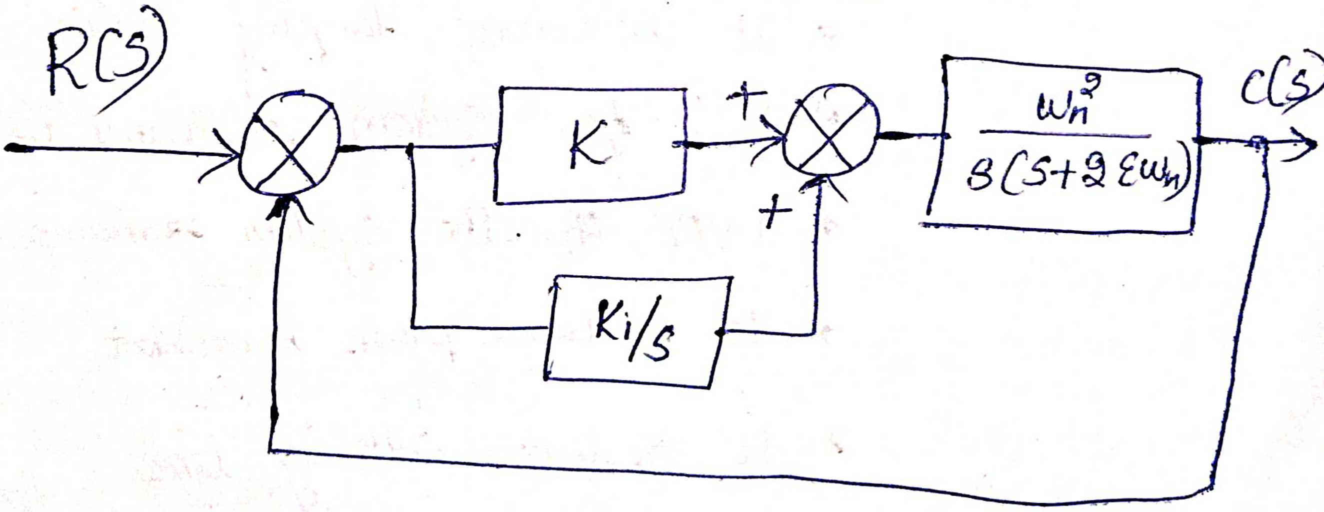
A Controller in the forward path, which changes the Controller output corresponding to the proportional plus integral of the error signal is called PI Controller.

$$\text{Output of Controller} = K\,e(t) + K_i \int e(t)\,dt$$

$$\text{Taking Laplace} = K\,E(S) + \frac{K_i}{S}E(S) = E(S)\left[K + \frac{K_i}{S}\right]$$

$$K = 1$$

$$G(S) = \frac{\left[1 + \frac{K_i}{S}\right]\omega_n^2}{S\left[S + 2\xi\omega_n\right]} = \frac{\left[K_i + S\right]\omega_n^2}{S^2\left[S + 2\xi\omega_n\right]}$$



$$\frac{C(S)}{R(S)} = \frac{(K_i + S)\omega_n^2}{S^3 + 2\xi\omega_n S^2 + S\omega_n^2 + K_i\omega_n^2}$$

$K=1$

$$G(S) = \frac{(1+ST_d)\,\omega_n^2}{S(S+2\xi\omega_n)}$$

$$\frac{C(S)}{R(S)} = \frac{(1+ST_d)\,\omega_n^2}{S^2+S[2\xi\omega_n+\omega_n^2 T_d]+\omega_n^2}$$

$$K_p = \lim_{S\to 0} G(S)\,H(S) = \infty$$

$$K_v = \lim_{S\to 0} S\,G(S)\,H(S) = \frac{\omega_n}{2\xi}$$

No change in coefficients error also will remain same.

Hence PD controller has following effects on system.

* It increases damping ratio.
* 'ω_n' for system remains unchanged
* 'TYPE' of the system remains unchanged
* It reduces peak overshoot
* It reduces settling time
* Steady state error remains unchanged.

it becomes third order

Transient response gets affected badly if controller is not designed properly.

$$K_p = \lim_{s \to 0} G(s) H(s) = \infty \quad , e_{ss} = 0$$

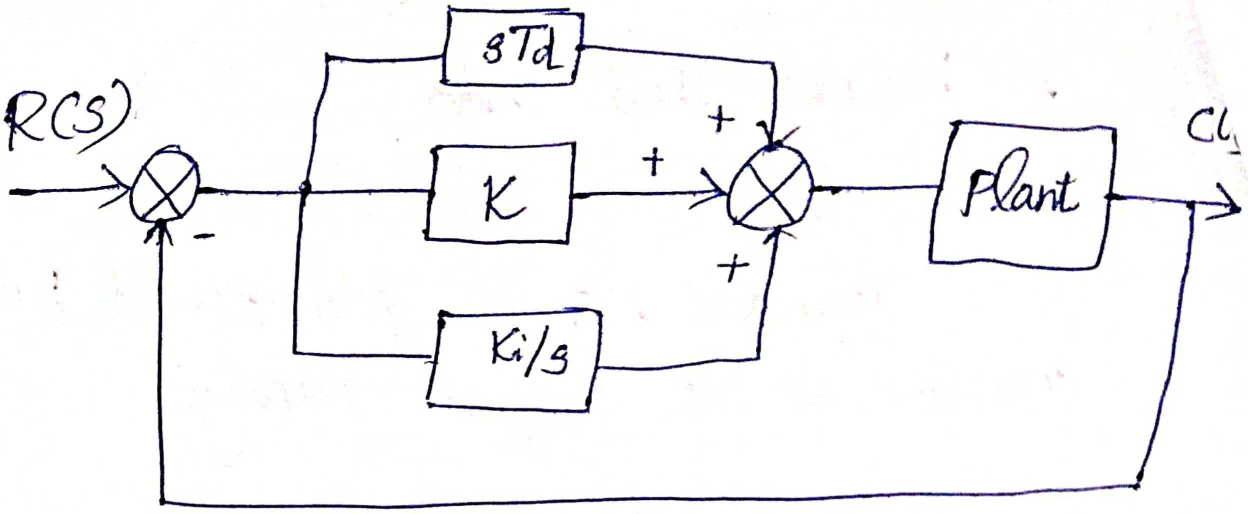
$$K_v = \lim_{s \to 0} s G(s) H(s) = \infty \quad , e_{ss} = 0.$$

PI Controller has following effects

1) It increases order of the system
2) It increases TYPE of the system
3) Steady state error reduces tremendously for same type of inputs

PID Type of controller

PD improves transient and PI improves steady state. Combination of two may be used to improve overall time response of the system.



Order and Type Number

Consider the system having the following function

$$G(S) = \frac{S^2+5}{S^2[S^2+3S+6]}$$

Order of the system is 4

Type of the system is 2

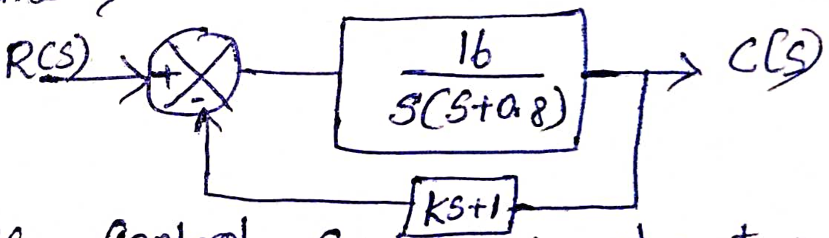
Order of the system:- The value of the highest exponent that appears in the denominator of the transfer function.

Type of the system: Number of poles at origin.

Assignment 2

1) The unity feedback system is characterized by an open loop transfer function $G(S) = \frac{K}{S(S+10)}$. Determine gain K. so that system will have damping ratio of 0.5 for this value of K. Determine settling time, Peak Overshoot and time at Peak overshoot for a unit step input.

2) A position control system with velocity feedback as shown. given that $\xi = 0.5$. Also calculate rise time, peak time, maximum overshoot and settling time.



3) A Unity feedback Control System is characterized by following open loop transfer function $G(S) = (0.4S+1)/S(S+0.6)$. Determine transient response for unit step input.

Steady error Constant and System.

The derivation of the output of Control System from desired response during Steady State is known as Steady State error. It is represented as e_{ss}.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s)$$

UNIT III

Frequency Response And System Analysis

Closed loop frequency response - Performance specification in frequency domain - Frequency response of second order System - Bode plot - Polar Plot - Nyquist plots - Design of Compensators using Bode plots - Cascade lead compensation - Cascade lag Compensation - Cascade lag-lead Compensation.

Introduction

The frequency response analysis deals with study of steady state response of the system to sinusoidal input of variable frequency.

Initially . frequency response analysis is the determination of system transfer function. Then it is expressed interms of magnitude & phase angle.

The sinusoidal transfer function is the frequency domain representation of the system (T(jw)), and so it is called frequency domain transfer function.

If the s domain transfer function, T(s) is known, then frequency domain transfer function T(jw) can be obtained directly from T(s) by replacing s by jw.

$$T(s) \xrightarrow{s=jw} T(jw)$$

$$T(jw) = |T(jw)| \angle T(jw)$$

$|T(jw)| \Rightarrow$ Magnitude of $T(jw)$

$\angle T(jw) \Rightarrow$ Phase of $T(jw)$

Closed loop frequency Response.

For a single loop control system configuration, closed loop transfer function is

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

under sinusoidal steady state $\boxed{s=j\omega}$, then

$$M(j\omega) = \frac{Y(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

Then sinusoidal steady state transfer function $M(j\omega)$ may be expressed in terms of its magnitude and phase.

$$M(j\omega) = |M(j\omega)| \angle M(j\omega)$$

Frequency Domain Specifications

The performance and characteristics of a system in frequency domain are measured interms of frequency domain specifications.

The frequency domain specifications are

1) Resonant Peak M_r
2) Resonant frequency, ω_r
3) Bandwidth, ω_b
4) Cut off rate
5) Gain Margin, K_g
6) Phase Margin, γ

1) Resonant Peak (M_r)

This is the maximum value of M and is denoted as M_r. The magnitude of resonance peak M_r provide us information about the relative stability of the system. Large resonant peak corresponds to Large overshoot in the transient response

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

(ii) Resonant frequency (ω_r)

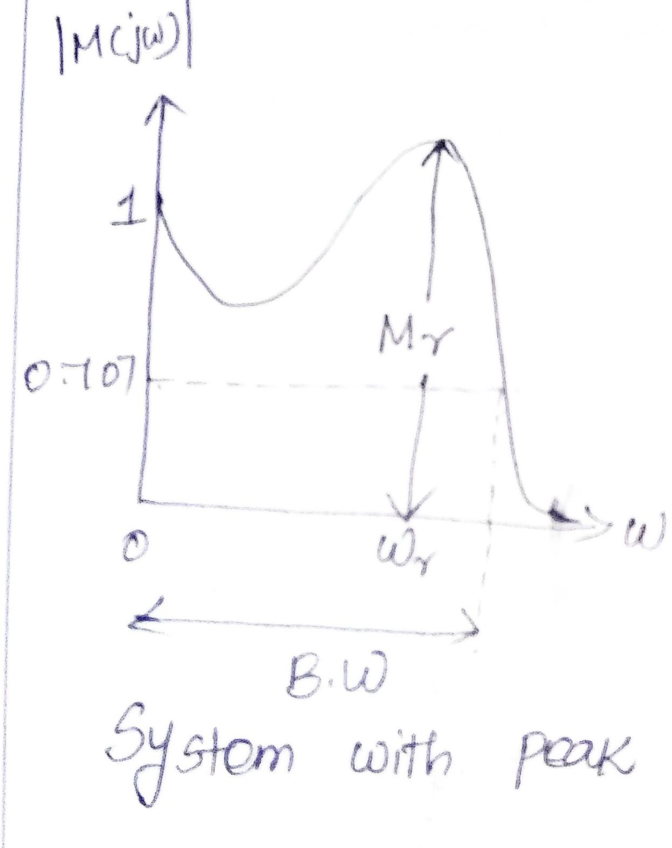
It is the frequency at which resonant peak occurs i.e., Maximum

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

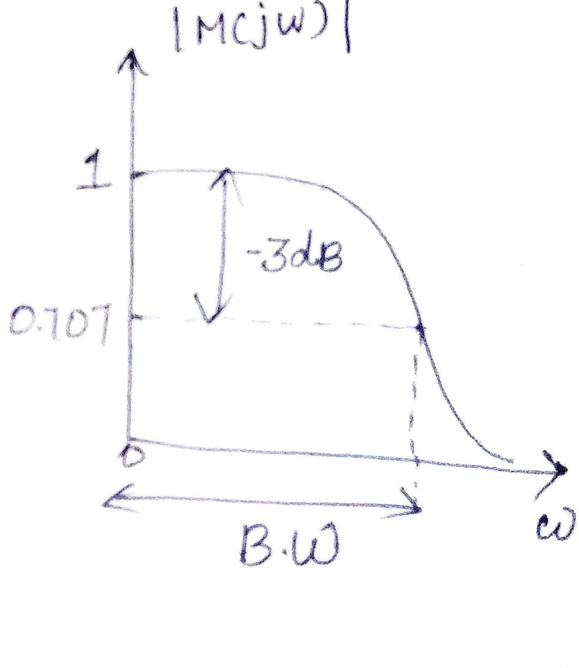
(iii) Band width (BW)

It is the frequency at which magnitude M is $\frac{1}{\sqrt{2}} = 0.707$ (or) $-3\,dB$ times its value at zero frequency.

$$B.W = \omega_n\sqrt{1-2\xi^2+\sqrt{2-4\xi^2+4\xi^4}}$$



System with peak



(iv) Cut off Rate

It is a Paramete which indicate the ability of a System in distinguishing Signals from noise. It is present at slope of $|M(j\omega)|$.

Apparently, if two systems have same Bandwidth, but cut off rates may be different.

(V) Gain Margin

The gain Margin is defined as the value of gain, to be added to system, in order to bring the system to the verge of instability.

$$GM = \frac{1}{|G(j\omega_\phi) H(j\omega_\phi)|}$$

gain cross over frequency (ω_g)

It is the frequency at which gain cross over the point.

$|G(j\omega_g) H(j\omega_g)| = 1$ (or) 0 dB [log 1 = 0 dB]

Phase Margin (PM) or (ϕ_M)

It is the angle by which phase $G(j\omega) H(j\omega)$ can be decreased to derive the system to verge of instability

$$\phi_M = 180^\circ + \angle G(j\omega_g) H(j\omega_g)$$

Phase cross over frequency (ω_ϕ)

It is the frequency at which phase cross over point.

$$\angle G(j\omega_\phi) H(j\omega_\phi) = -180^\circ$$

1) The forward path transfer function of a unity feedback control system is given as

$$G(s) = \frac{64}{s(s+5)}$$

calculate resonant peak, resonant frequency and Bandwidth of closed loop system.

Solution

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{G(s)}{1+G(s)} \qquad (\because H(s)=1)$$

$$= \frac{\frac{64}{s(s+5)}}{1+\frac{64}{s(s+5)}} = \frac{64}{s(s+5)} \times \frac{s(s+5)}{s^2+5s+64}$$

$$\frac{C(s)}{R(s)} = \frac{64}{s^2+5s+64}$$

Compare with general second order transfer function

$$\omega_n^2 = 64 \Rightarrow \omega_n = 8$$

$$2\xi\omega_n = 5 \Rightarrow \xi = 0.3125$$

Resonant frequency $\omega_r = \omega_n\sqrt{1-2\xi^2}$

$$= 8\sqrt{1-2(0.3125)^2}$$

$$\boxed{\omega_r = 7.1 \text{ rad/sec}}$$

Resonant Peak $= M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$

$$= \frac{1}{2(0.3125)\sqrt{1-0.3125^2}}$$

$$\boxed{M_r = 1.78}$$

Bandwidth $\omega_b = \omega_n\sqrt{1-2\xi^2+(2-4\xi^2+4\xi^4)^{1/2}}$

$$\boxed{\omega_b = 11.25 \text{ rad/sec}}$$

2) Find the Bandwidth of the System whose transfer function is $1/s+1$

Solution

$$T(s) = \frac{1}{s+1}$$

$$T(j\omega) = \frac{1}{j\omega+1}$$

$$T(j\omega) = \frac{1}{1+j\omega}$$

$$M = \frac{1}{\sqrt{1+\omega^2}}$$

In dB, $M = 20\log\left(\frac{1}{\sqrt{1+\omega^2}}\right) = 20\log\left(\sqrt{1+\omega^2}\right)^{-1}$

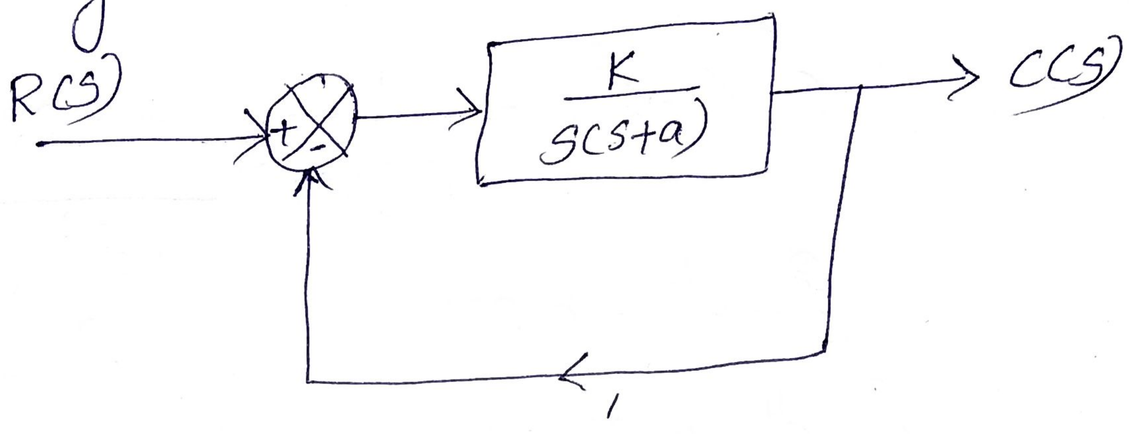
$$= -20\log\left(\sqrt{1+\omega^2}\right)$$

for $\omega = 0$, $M = 0dB$

$\omega = \omega_b$, $M = -3dB \Rightarrow -3 = -20\log\sqrt{1+\omega_b}$

$$\boxed{\omega_b = 0.9976 \text{ rad/sec}}$$

3) For a given system, determine the value of K and a, to satisfy the following frequency domain specifications: $M_r = 1.04$, $\omega_r = 11.55$ rad/sec, For the values of K and a, calculate the settling time and Bandwidth of the system.



Solution

$$G(S) = \frac{K}{S(S+a)} , \quad H(S) = 1$$

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{K/S(S+a)}{1+K/S(S+a)}$$

$$= \frac{K}{S^2+aS+K}$$

Compare with second order T.F

$$\omega_n^2 = K \Rightarrow \omega_n = \sqrt{K} \text{ rad/sec}$$

$$2\xi\omega_n = a \Rightarrow \xi = \frac{a}{2\sqrt{K}}.$$

Given

$M_r = 1.04, \quad \omega_r = 11.55$ rad/sec

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

Squarring on both sides

$$M_r^2 = \frac{1}{4\xi^2(1-\xi^2)}$$

$$\xi^2(1-\xi^2) = \frac{1}{4(1.04)^2} = 0.231$$

$$\xi^2 - \xi^4 = 0.231$$

$$\xi^2 - \xi^4 - 0.231 = 0$$

$$-\left[\xi^4 - \xi^2 + 0.231\right] = 0.$$

$$(\xi^2)^2 - \xi^2 + 0.231 = 0.$$

$$\xi^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1 \quad b = -1 \quad c = 0.231$

$$\xi^2 = \frac{1 \pm \sqrt{0.076}}{2}$$

roots $\xi^2 = \frac{1+\sqrt{0.076}}{2}, \frac{1-\sqrt{0.076}}{2}$

$$\boxed{\xi^2 = 0.6373, 0.3676}$$

$\boxed{\xi = 0.6021}$ for $\xi > 0.707$ for which M_r does not exist

$\omega_r = 11.55$

$\xi^2 = 0.3676$

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

$$\boxed{\omega_n = 22.033 \text{ rad/sec}}$$

$$\boxed{K = \omega_n^2 = 485.453}$$

$$\xi = \frac{a}{2\sqrt{K}}$$

$$a = 2 \times 0.6021 \times 22.033$$

$$\boxed{a = 26.5321}$$

$$T_s = \frac{4}{\xi \omega_n} = 0.3015 \text{ sec}$$

$$B.W = \omega_n \sqrt{1 - 2\xi^2 + \sqrt{2 - 4\xi^2 + 4\xi^4}}$$

$$\boxed{B.W = 25.237 \text{ rad/sec}}$$

Frequency response analysis of control systems can be carried either analytically or graphically. The various graphical techniques available for frequency response analysis are,

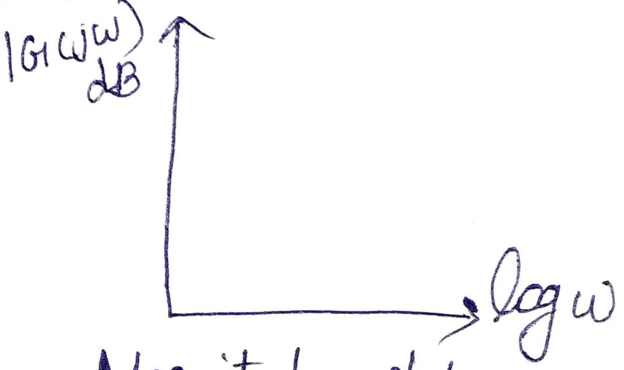
(i) Bode plot
(ii) polar plot
(iii) Nyquist plot.

Bode Plot

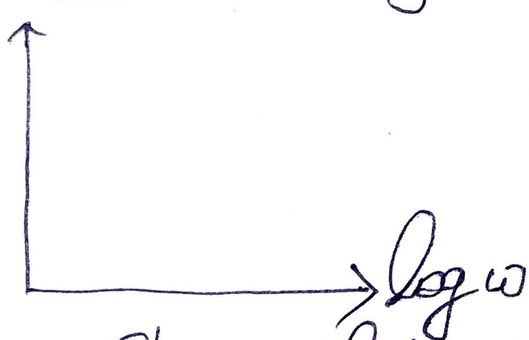
It is named after Hendrick. W. Bode. Bode plot is one of the powerful graphical methods of analyzing and designing control systems. It consists of two graphs.

(i) plot logarithm of magnitude of sinusoidal transfer function Vs frequency in logarithmic scale.

(ii) Plot phase angles in degrees Vs logarithmic frequency



Magnitude plot



Phase plot

1) Sketch Bode plot for the following transfer function and determine the system gain K for the gain cross over frequency to be 5 rad/sec.

$$G(S) = \frac{KS^2}{(1+0.2S)(1+0.02S)}$$

Solution:-

$$T.F = G(S) = \frac{(KS)^2}{(1+0.2S)(1+0.02S)}$$

Replace 's' by 'jw'

$$T.F = G(jw) = \frac{K(jw)^2}{(1+0.2jw)(1+0.02jw)}$$

Let K=1,

$$\boxed{G(jw) = \frac{(jw)^2}{(1+j0.2w)(1+j0.02w)}}$$

Step 1:- Finding out Corner frequencies

$$\text{Corner freq} = \frac{\text{Real Value}}{\text{Imaginary value}}$$

$$W_{c1} = \frac{1}{0.2} = 5 \text{ rad/sec}$$

$$W_{c2} = \frac{1}{0.02} = 50 \text{ rad/sec.}$$

Step 2:- Finding out change in slope at Corner frequency.

S.no	Term	Corner freq rad/sec	slop dB/dec	change in Slope (dB/dec)
1	$(jw)^2$	-	40	
2	$\frac{1}{1+j0.2w}$	$W_{c1} = 5$	-20	$40-20=20$
3	$\frac{1}{1+j0.02w}$	$W_{c2} = 50$	-20	$20-20=0$

Step 3:- choosing low frequency (ω_L) & high frequency (W_H).

choose a low frequency ω_L such that $\boxed{\omega_L < W_{c1}}$

choose high frequency W_H such that $\boxed{W_H > W_{c2}}$

Assume, $W_l = 0.5$ rad/sec & $W_H = 100$ rad/sec

Step 4:. Then calculate the gain value.

Gain at w_y = change in gain from w_x to w_y + Gain at w_x

$= \left[\text{slope from } w_x \text{ to } w_y \times \log \frac{w_y}{w_x}\right] +$ Gain at w_x

$A = |G(jw)|$ in dB

At $w = w_l$, $A = 20 \log |(jw)^2| = 20 \log (w)^2 = 20 \log (0.5)^2$
$= -12$ db

At $w = w_{c_1}$, $A = 20 \log |(jw)^2| = 20 \log (w)^2 = 20 \log (5)^2$
$= 28$ db

At $w = w_{c_2}$, $A = \left[\text{slope from } w_{c_1} \text{ to } w_{c_2} \times \log \frac{w_{c_2}}{w_{c_1}}\right] + A_{(\text{at } w = w_{c_1})}$

$= \left[20 \times \log \frac{50}{5}\right] + 28$

$= 48$ dB.

At $w = w_h$, $A = \left[\text{slope from } w_{c_2} \text{ to } w_h \times \log \frac{w_h}{w_{c_2}}\right] +$ A (at $w = w_{c_2}$)

$= \left[0 \times \log \frac{100}{50}\right] + 48 = 48$ dB.

Magnitude Values

Points in graph

$W_l = -12$ at 0.5 rad/sec $\Rightarrow$ a

$W_{c1} = 28$ at 5 rad/sec $\Rightarrow$ b

$W_{c2} = 48$ at 50 rad/sec $\Rightarrow$ c

$W_h = 48$ at 100 rad/sec $\Rightarrow$ d

Join the points by straight line in semilog graph paper and mark the slope on respective region.

Step 5:- Calculate phase angle.

$$\phi = \angle G(j\omega) = 180^\circ - \tan^{-1}\left(\frac{0.2\omega}{1}\right) - \tan^{-1}\left(\frac{0.02\omega}{1}\right)$$

S.no	ω rad/sec $(\omega \times \frac{180}{\pi})$ deg	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} 0.02\omega$ deg	$\angle\phi = \angle G(j\omega)$ deg	Points in graph
1	0.5	5.7	0.6	174	e
2	1	11.3	1.1	168	f
3	5	45	5.7	130	g
4	10	63.4	11.3	106	h
5	50	84.3	45	50	i
6	100	87.1	63.4	30	j

plot the phase points in semilog sheet and join the points by smooth curve.

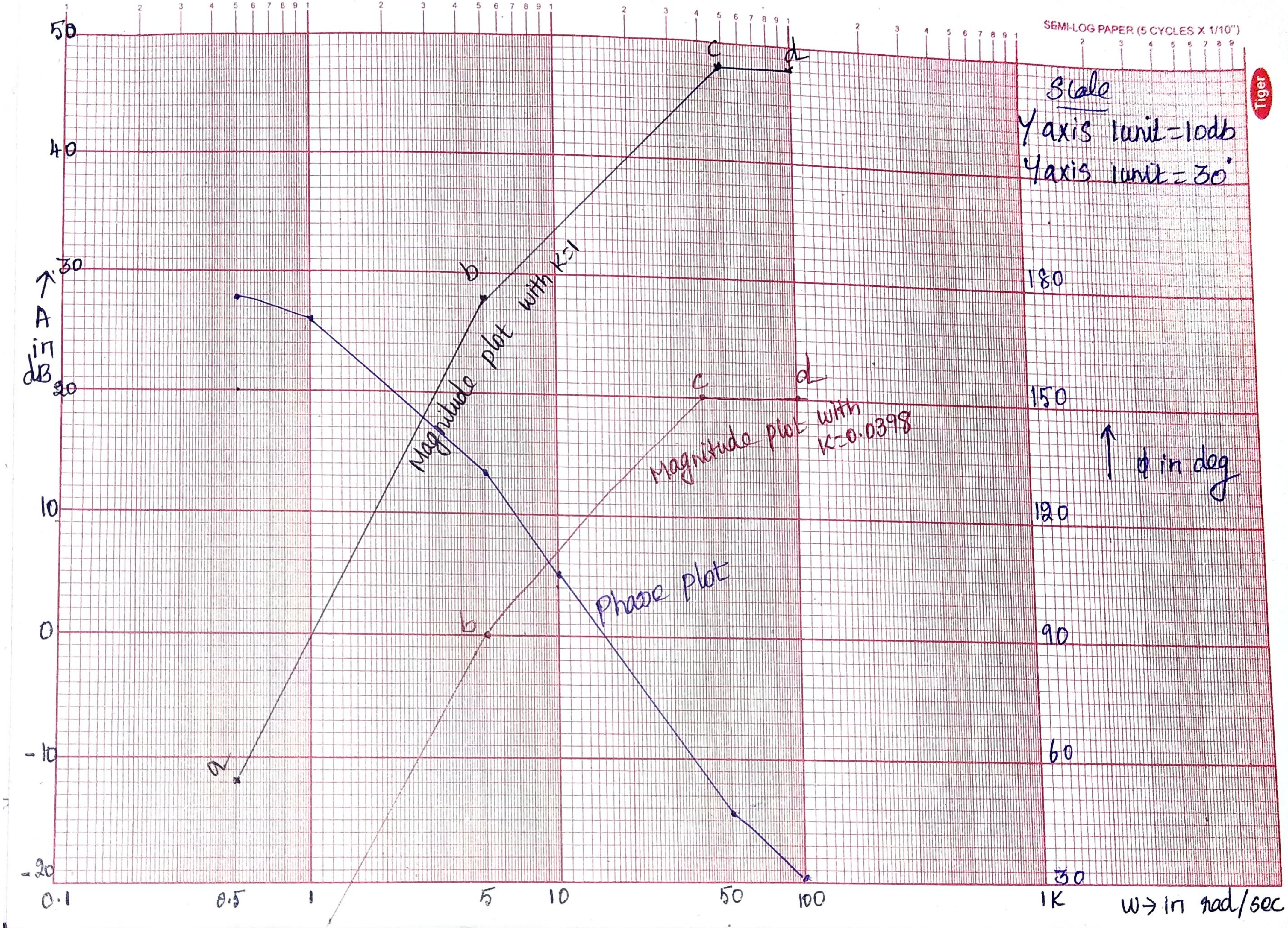
Step 5 :- Calculation of K.

Given that gain cross over frequency is 5 rad/sec At $w=5$ rad/sec, gain is 28 db. Hence to every point of magnitude plot of dB the gain of -28 dB should be added. The addition of -28 dB shift plot downwards

$$20 \log K = -28 dB$$

$$\log K = \frac{-28}{20}$$

$\boxed{K = 0.0398}$ ⇒ Corresponding magnitude plot is plotted in semilog graph


SEMI-LOG PAPER (5 CYCLES X 1/10")
Tiger
Scale
Y axis 1unit = 10db
Y axis 1unit = 30°
A in dB
ϕ in deg
Magnitude plot with K=1
Magnitude plot with K=0.0398
Phase plot
a
b
c
d
b
c
d
50
40
30
20
10
0
-10
-20
180
150
120
90
60
30
0.1
0.5
1
5
10
50
100
1K
w → in rad/sec

2) Sketch the Bode plot for following transfer function and determine phase margin and gain Margin.

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)}$$

Solution

By Comparing G(s) with second order T.F

$\omega_n^2 = 100$ $\qquad\qquad$ $2\xi\omega_n = 16$

$\omega_n = 10$ rad/sec $\qquad\qquad$ $\xi = \frac{16}{2\omega_n} = 0.8$

to Convert s domain to Bode form

$$G(s) = \frac{75(1+0.2s)}{s(s^2+16s+100)} = \frac{\frac{75}{100}(1+0.2s)}{s\left(\frac{s^2}{100}+\frac{16}{100}s+\frac{100}{100}\right)}$$

$$G(s) = \frac{0.75(1+0.2s)}{s(1+0.16s+0.01s^2)}$$

Put $s = j\omega$

$$G(j\omega) = \frac{0.75(1+0.2j\omega)}{j\omega[1+0.16j\omega+0.01(j\omega)^2]} = \frac{0.75(1+0.2j\omega)}{j\omega[1-0.01\omega^2+j0.16\omega]}$$

Step 1 :- Magnitude plot

Corner frequencies $\omega_{c_1} = \frac{1}{0.2} = 5$ rad/sec

$\omega_{c_2} = \omega_n = 10$ rad/sec

because, for Quadratic factor, Corner freq $= \omega_n$

S.No	Term	Corner freq rad/sec	Slope dB/sec	change in slope dB/dec
1	$\frac{0.75}{j\omega}$	-	-20	
2	$1+j0.2\omega$	$\omega_{c_1} = 5$	20	$-20+20=0$
3.	$1-0.01\omega^2+j0.16\omega$	$\omega_{c_2}=\omega_n=10$	-40	$0-40=-40$

Step 2:

$\omega_l = 0.5$ rad/sec $\qquad \omega_h = 20$ rad/sec

At $\omega=\omega_l \Rightarrow A = 20\log\left|\frac{0.75}{\omega}\right| = 3.5$ dB

$\omega=\omega_{c_1} \Rightarrow A = 20\log\left|\frac{0.75}{\omega}\right) = -16.5$ dB

$\omega = \omega_{c_2} \Rightarrow A = \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\frac{\omega_{c_2}}{\omega_{c_1}}\right] +$ A at $\omega=\omega_{c_1}$

$= \left(0 \times \log\frac{10}{5}\right) - 16.5$

$= -16.5$ dB

$\omega = \omega_h \Rightarrow A = \left[(\text{slope from } \omega_{c2} \text{ to } \omega_h) \times \log \frac{\omega_h}{\omega_{c2}} \right]$
$+$
$A_{at\ \omega = \omega_{c2}}$

$= \left(-40 \times \log \frac{20}{10}\right) - 16.5$

$= -28.5 \text{ dB.}$

S.No	ω	ω Value	Magnitude Values	Points in graph
1	ω_l	0.5	3.5	a
2	ω_{c1}	5	−16.5	b
3	ω_{c2}	10	−16.5	c
4	ω_h	20	−28.5	d

Step 3: phase plot

$$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90^\circ - \tan^{-1}\left(\frac{0.16\omega}{1 - 0.01\omega^2} + 180^\circ\right)$$

for $\omega > \omega_n$

$$\phi = \angle G(j\omega) = \tan^{-1} 0.2\omega - 90^\circ - \tan^{-1}\left(\frac{0.16\omega}{0.01\omega^2}\right)$$

for $\omega \leq \omega_n$

S.No	ω rad/sec	$\tan^{-1} 0.2\omega$ deg	$\tan^{-1} \frac{0.16\omega}{1-0.01\omega^2}$ deg	$\phi = \angle G(j\omega)$	points
1	0.5	5.7	4.6	−88	e
2	1	11.3	9.2	−88	f
3	5	45	46.8	−92	g
4	10	63.4	90	−116	h
5	20	75.9	−46.8+180 = 133.2	−148	i
6	50	84.3	−18.4 +180 = 161.6	−168	j
7	100	87.1	−92+180 = 170.8	−174	K

Let ϕ_{gc} be phase of $G(j\omega)$ at gain cross over frequency, ω_{gc}. from graph we get,

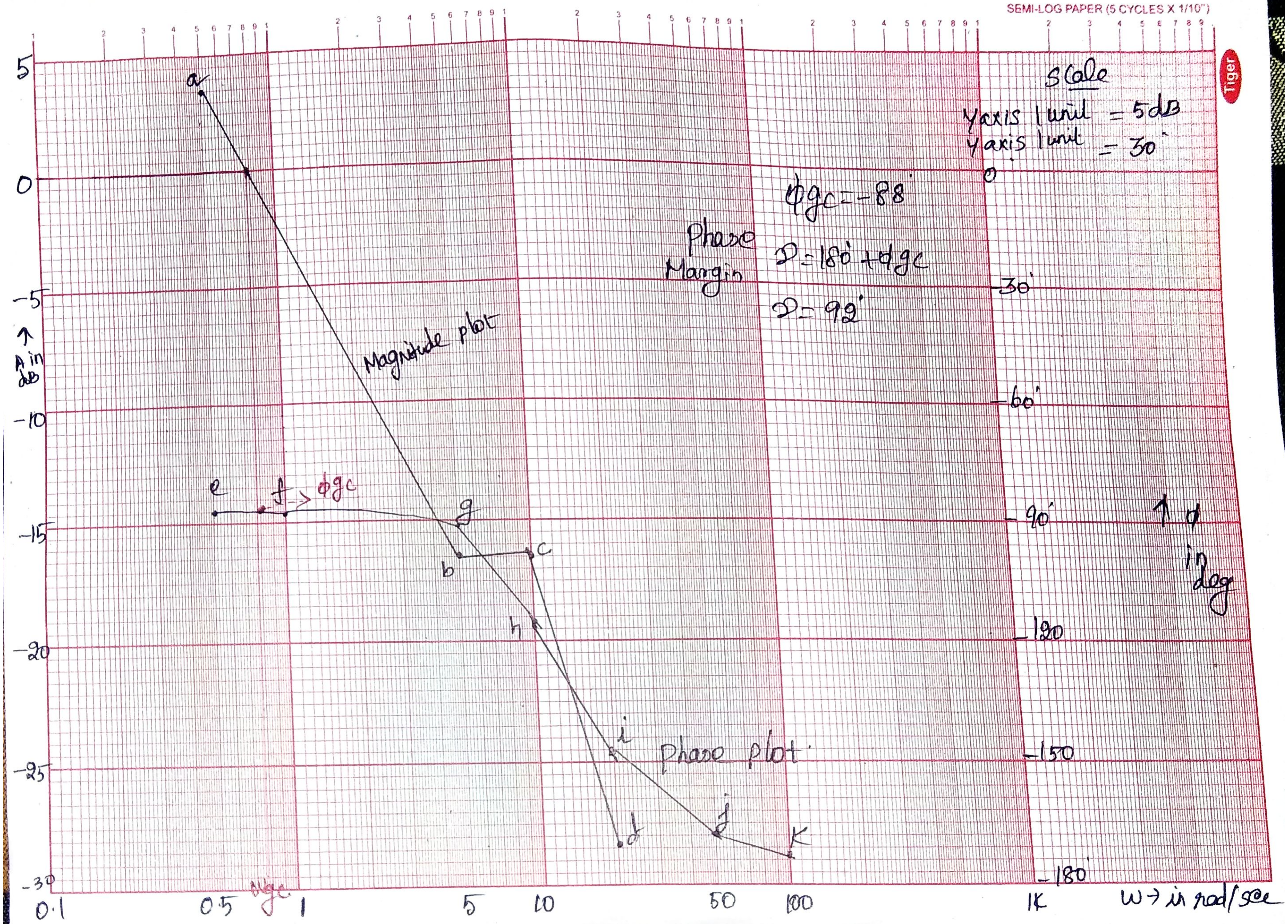
$$\phi_{gc} = -88^\circ$$

Phase margin, $\gamma = 180^\circ + \phi_{gc} = 180^\circ - 88 = 92^\circ$

The phase plot crosses -180° only at infinity

$|G(j\omega)|$ at infinity $= -\infty$ dB

Hence gain Margin $= +\infty$.



SEMI-LOG PAPER (5 CYCLES X 1/10")
Tiger
Scale
Y axis 1 unit = 5 dB
Y axis 1 unit = 30°
Φgc = -88°
Phase Margin
γ = 180° + Φgc
γ = 92°
Magnitude plot
Phase plot
A in dB
Φ in deg
f → Φgc
Φgc
ω → in rad/sec
5
0
-5
-10
-15
-20
-25
-30
0
-30°
-60°
-90°
-120
-150
-180°
0.1
0.5
1
5
10
50
100
1K
a
b
c
d
e
f
g
h
i
j
K

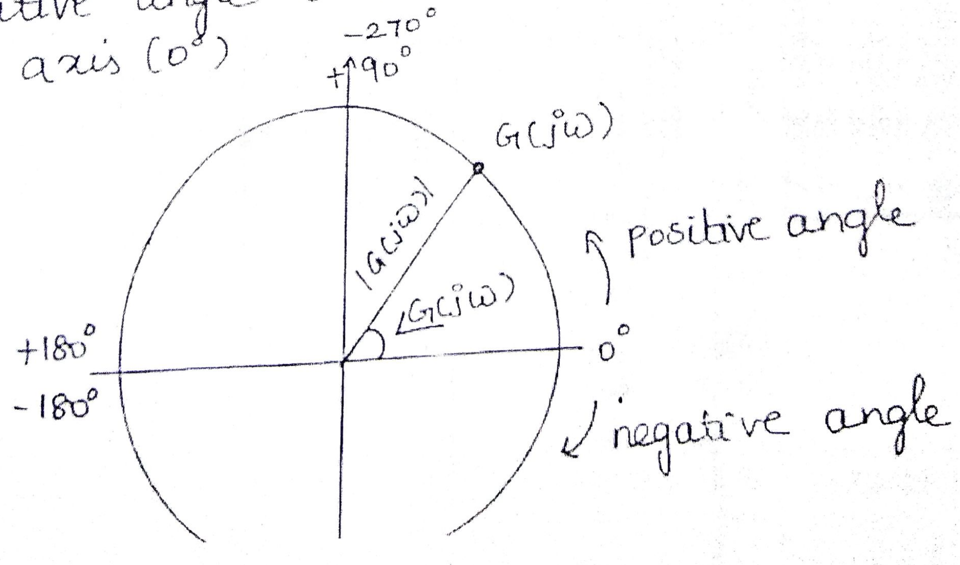
POLAR PLOT

The polar plot of sinusoidal transfer function $G(j\omega)$ is a plot of magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity. Thus polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity.

Polar plot is the base of Nyquist plot and the stability analysis using Nyquist plot method. It is not necessary to convert magnitude to its dB value or find logarithm of frequencies.

Polar plot is usually plotted on polar graph sheet. This sheet has concentric circles and radial lines. concentric circles represent magnitude and radial lines represent phase angles.

In that sheet, positive phase angle is measured in anticlockwise from reference axis (0°) and negative angle is measured clockwise from reference axis (0°)

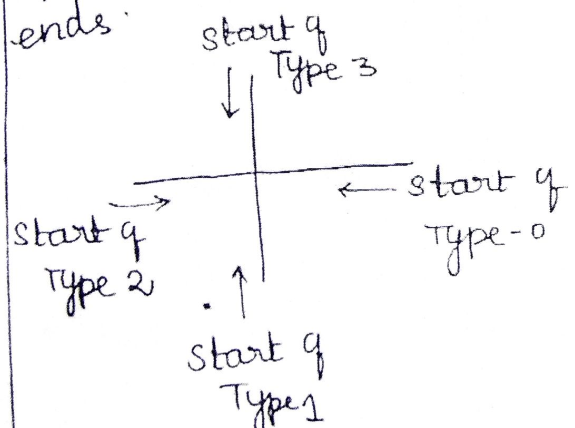


Alternative if $G(j\omega)$ can be expressed in rectangular coordinates,

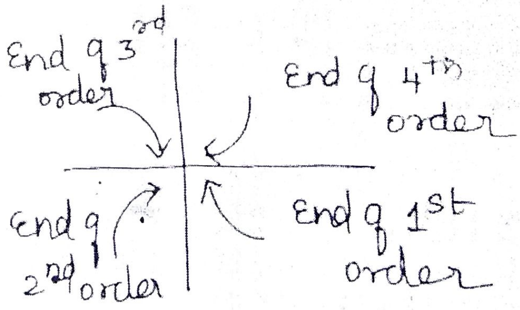
$$G(j\omega) = \underbrace{G_R(j\omega)}_{\text{Real part of } G(j\omega)} + j\,\underbrace{G_I(j\omega)}_{\text{Img. part of } G(j\omega)}$$

Then, polar plot can be plotted in ordinary graph sheet between $G_R(j\omega)$ and $G_I(j\omega)$ by varying ω from 0 to ∞. In order to plot the polar plot on ordinary graph sheet, magnitude & phase of $G(j\omega)$ are computed for various values of ω. Then convert polar coordinates to rectangular coordinates using P→R in calculator.

For minimum phase transfer function with only poles, type number of the system determines the quadrant at which polar plot starts and order of system determines quadrant at which polar plot ends.



Start of polar plot



End of polar plot

The change in shape of polar plot can be predicted due to addition of pole or zero.

(i) when a pole is added to a system, polar plot end point will shift by -90°

(ii) when a zero is added to a system, polar plot end point will shift by $+90^\circ$

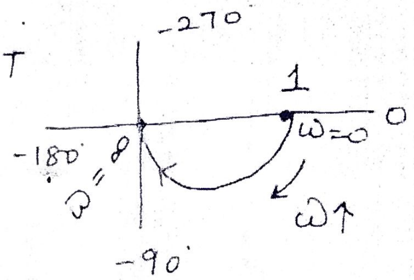
Typical sketches of polar plot.

(i) $\boxed{G(s) = \frac{1}{1+sT}}$

$\Rightarrow$ Type: 0, order: 1

$$G(j\omega) = \frac{1}{1+j\omega T} = \frac{1}{\sqrt{1+\omega^2T^2} \angle \tan^{-1}\omega T}$$

$$G(j\omega) = \frac{1}{\sqrt{1+\omega^2T^2}} \angle -\tan^{-1}\omega T.$$



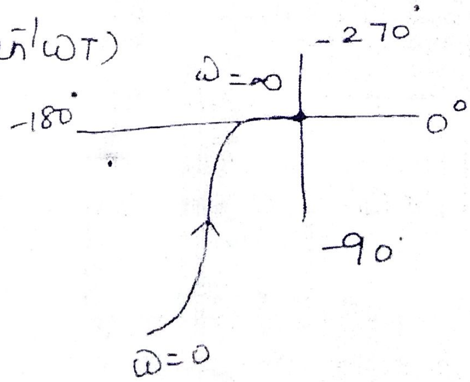
$\omega \to 0$, $G(j\omega) = 1\angle 0^\circ$

$\omega \to \infty$, $G(j\omega) = 0\angle -90^\circ$

(ii) $\boxed{G(s) = \frac{1}{s(1+sT)}}$ $= \frac{1}{j\omega(1+j\omega T)} = \frac{1}{\omega \angle 90^\circ \sqrt{1+\omega^2T^2} \angle \tan^{-1}\omega T}$

$\Rightarrow$ Type: 1, order: 2

$$G(j\omega) = \frac{1}{\omega\sqrt{1+\omega^2T^2}} \angle(-90^\circ - \tan^{-1}\omega T)$$



$\omega = 0$, $G(j\omega) = \infty \angle -90^\circ$

$\omega = \infty$, $G(j\omega) = 0\angle -180^\circ$

(iii) $\boxed{G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)}}$

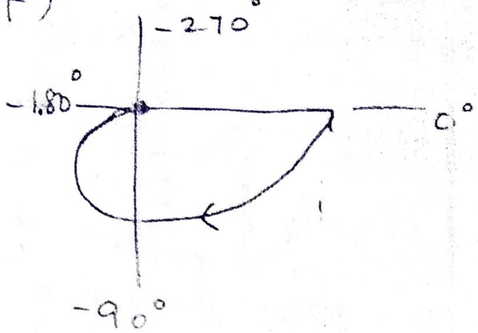
Type: 0, order: 2

$$G(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{\sqrt{1+\omega^2T_1^2}\angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2T_2^2}\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega\sqrt{1+\omega^2T^2}} \angle(-90^\circ - \tan^{-1}\omega T)$$

As $\omega \to 0$ $G(j\omega) = 1\angle 0^\circ$

$\omega \to \infty$. $G(j\omega) = 0\angle -180^\circ$



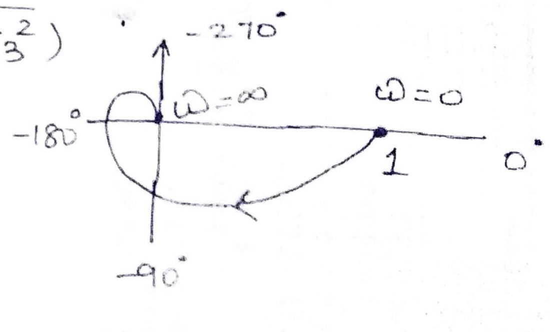
(iv) $\boxed{G(s) = \dfrac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}}$

Type: 0, order: 3

$$G(j\omega) = \frac{\angle -\tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3}{\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}}$$

As $\omega \to 0$, $G(j\omega) = 1\angle 0^\circ$

$\omega \to \infty$, $G(j\omega) = 0\angle -270^\circ$

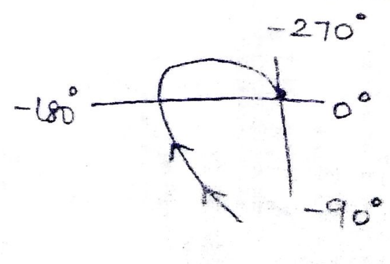


(v) $\boxed{G(s) = \dfrac{1}{s(1+sT_1)(1+sT_2)}}$

Type: 1; order: 3

$$G(j\omega) = \frac{1}{\omega\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}} \angle -90^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2$$

$\omega = 0$, $G(j\omega) = \infty \angle -90^\circ$



.i) $\boxed{G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)}}$

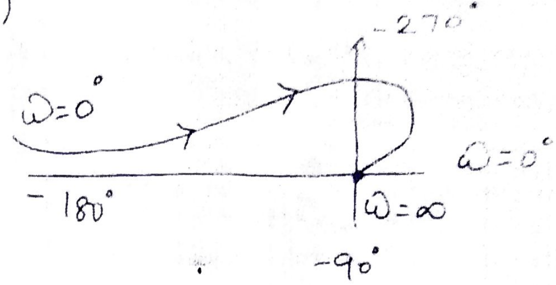
Type: 2, order: 4

$$G(j\omega) = \frac{1}{\omega^2 \angle -180^\circ \sqrt{1+\omega^2 T_1^2}\,\angle \tan^{-1}\omega T_1 \sqrt{1+\omega^2 T_2^2}\,\angle \tan^{-1}\omega T_2}$$

$$= \frac{1}{\omega^2 \sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)}} \angle(-180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2)$$

$\omega = 0$, $G(j\omega) = \infty \angle -180^\circ$

$\omega = \infty$, $G(j\omega) = 0 \angle -360^\circ$



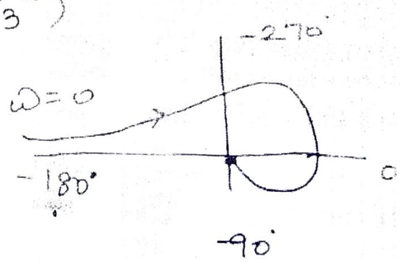
(vii) $\boxed{G(s) = \dfrac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}}$

Type: 2, order: 5

$$G(j\omega) = \frac{1}{\omega^2 \sqrt{(1+\omega^2 T_1^2)(1+\omega^2 T_2^2)(1+\omega^2 T_3^2)}} \angle -180^\circ - \tan^{-1}\omega T_1 - \tan^{-1}\omega T_2 - \tan^{-1}\omega T_3$$

$\omega = 0$, $G(j\omega) = \infty \angle -180^\circ$

$\omega = \infty$, $G(j\omega) = 0 \angle -450^\circ = 0 \angle -90^\circ$

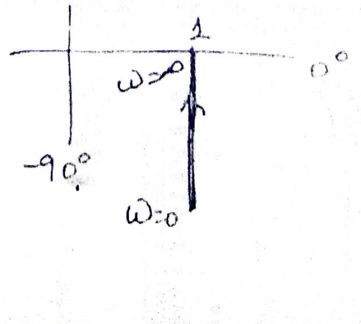


(viii) $\boxed{G(s) = \dfrac{1+sT}{sT}}$

$$G(j\omega) = \frac{1+j\omega T}{j\omega T} = \frac{1}{j\omega T + 1} = \frac{1}{\omega T \angle 90^\circ} + 1 = \frac{1}{\omega T} \angle -90^\circ + 1$$

$\omega = 0$, $G(j\omega) = \infty \angle -90^\circ + 1$

$\omega = \infty$, $G(j\omega) = 0 \angle -90^\circ + 1$



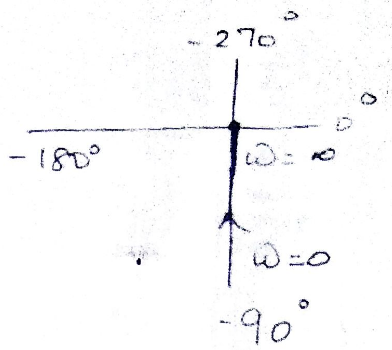
(ix) $\boxed{G(s) = 1/s}$

$\Rightarrow$ Type: 1, order: 1

$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} \angle -90^\circ$

At $\omega = 0$, $G(j\omega) = \infty \angle -90^\circ$

At $\omega = +\infty$, $G(j\infty) = 0 \angle -90^\circ$



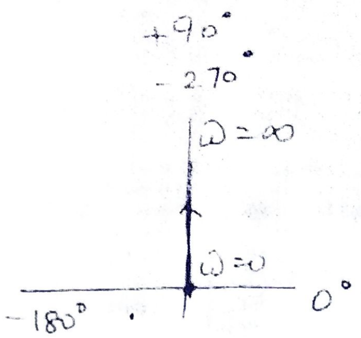
x) $\boxed{G(s) = s}$

$= \quad G(s) = s$

$= G(j\omega) = j\omega = \omega \angle 90^\circ$

At $\omega = 0$, $G(j\omega) = 0 \angle 90^\circ$

At $\omega = \infty$, $G(j\omega) = \infty \angle 90^\circ$



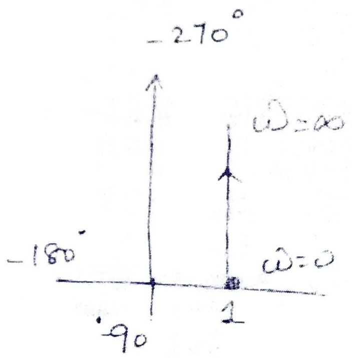
xi) $\boxed{G(s) = 1 + sT}$

$G(j\omega) = 1 + j\omega T$

$G(j\omega) = 1 + \omega T \angle 90^\circ$

At $\omega = 0$, $G(j\omega) = 1 \angle 90^\circ$

$\omega = \infty$, $G(j\omega) = 1 + \infty \angle 90^\circ$



— X —

1. The open loop transfer function of a unity feedback system is given by $G(S) = 1/S(1+S)(1+2S)$ sketch polar plot and determine gain margin and phase margin

= solution :-

$$G(S) = \frac{1}{S(1+S)(1+2S)}.$$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j2\omega)}$$

corner frequencies = $\omega_{c1} = \frac{1}{2} = 0.5$ rad/sec

$\omega_{c2} = 1$ rad/sec.

$\omega_l < \omega_{c1}$ & $\omega_h > \omega_{c2}$.

Phase angle :-

$$G(j\omega) = \frac{1}{\omega\sqrt{(1+\omega^2)(1+4\omega^2)}} \angle -90^\circ - \tan^{-1}\omega - \tan^{-1}2\omega$$

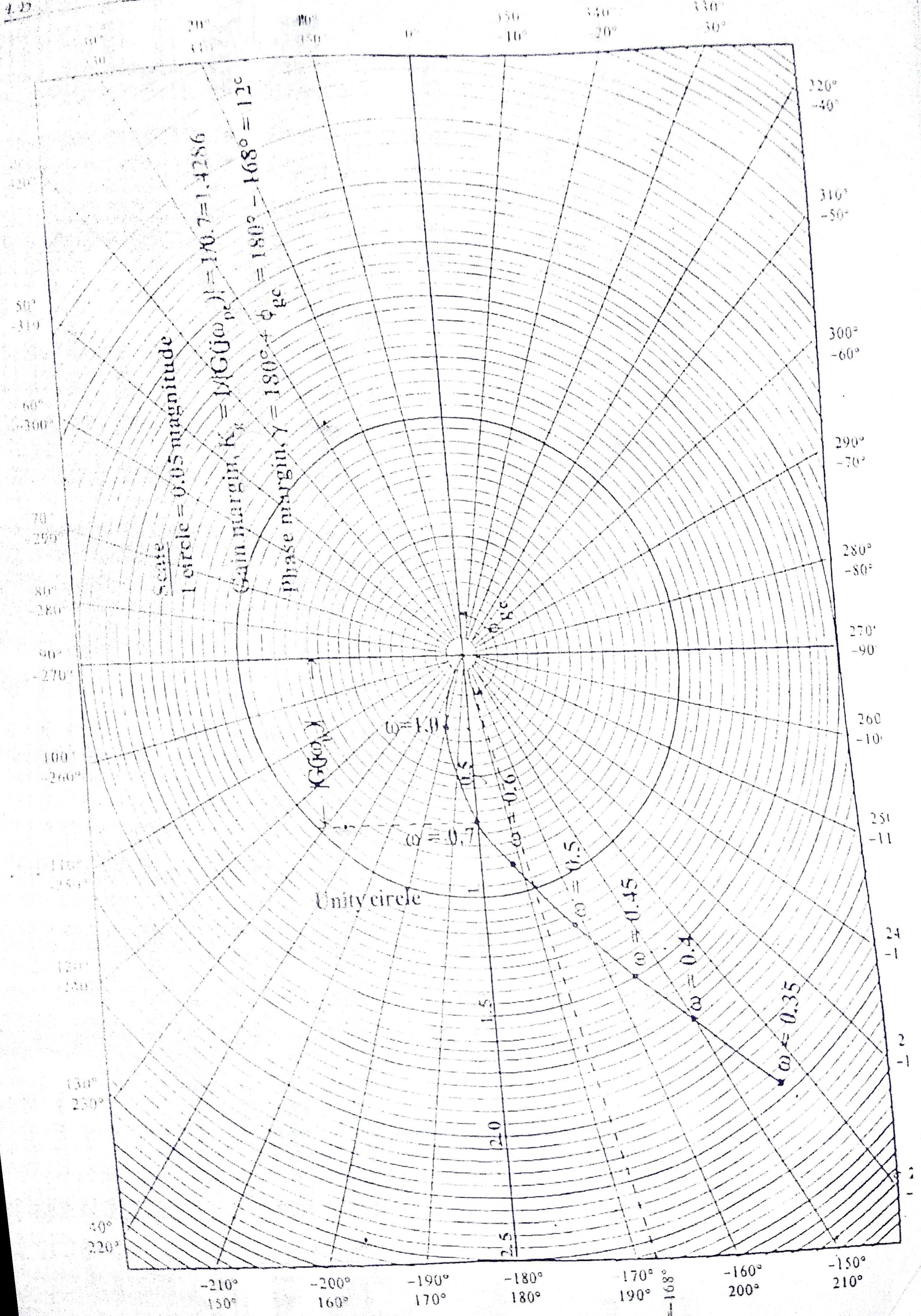
Magnitude & phase of $G(j\omega)$ at various freq.

ω	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.9	0.7	0.3
$\angle G(j\omega)$	-144	-150	-156	-162	-171	-180°	-198

Real & Imaginary part of $G(j\omega)$ at various freq

ω	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	-1.78	-1.56	-1.37	-1.14	-0.89	-0.7	-0.29
$G_I(j\omega)$	-1.29	-0.9	-0.61	-0.37	-0.14	0	0.09

Result: Gain Margin : $K_g = 1.4286$; Phase margin $\gamma = +12^\circ$


Scale
1 circle = 0.05 magnitude
Gain margin, $K_g = 1/|G(j\omega_{pc})| = 1/0.7 = 1.4286$
Phase margin, $\gamma = 180° + \phi_{gc} = 180° - 168° = 12°$
Unity circle
ω = 1.0
ω = 0.7
ω = 0.6
ω = 0.5
ω = 0.45
ω = 0.4
ω = 0.35
-210° 150°
-200° 160°
-190° 170°
-180° 180°
170° 190°
-160° 200°
-150° 210°
320° -40°
310° -50°
300° -60°
290° -70°
280° -80°
270° -90°
30° 330°
20° 340°
10° 350°
0°

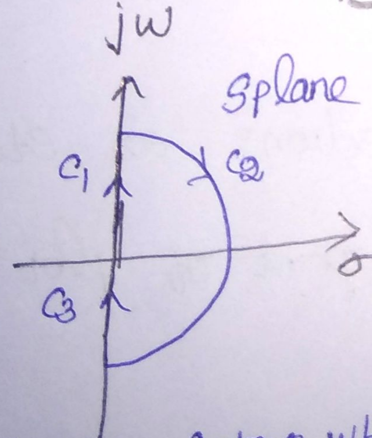
Nyquist plots

Nyquist plots are the Continuation of polar plots for finding the stability of the closed loop control systems by varying ω from −∞ to ∞.

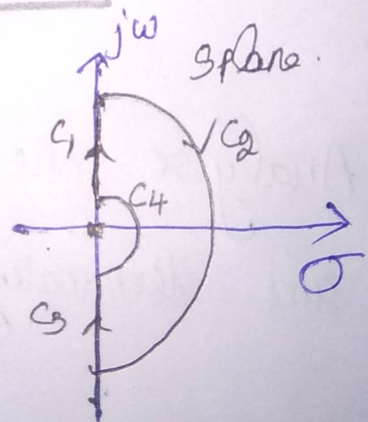
Nyquist Stability Criterion

It states that if there are P poles and Z zeros are enclosed by the 's' plane closed path, then the corresponding G(S) H(S) plane must encircle the origin Z-P times. So we can write the number of encirclement N as

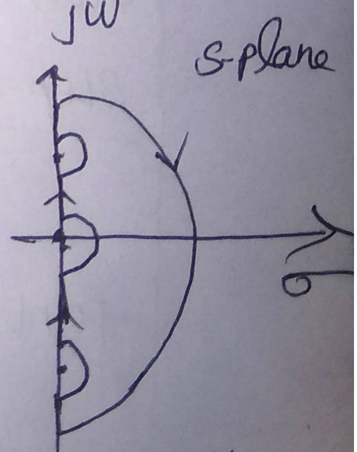
$$\boxed{N = Z - P}$$



Nyquist Contour when there is no pole on imaginary axis.



When there are poles at origin



when there are poles on imaginary axis. and at origin

Steps to solve Problems by Nyquist Criteria

Step 1:-

Count how many number of poles of $G(s) H(s)$ are in the right half of s plane (i.e) positive real part. This is the value of P.

Step 2:

Decide the stability Criterion on as $N = -P$ how many times Nyquist plot should encircle $-1+j0$ point for absolute stability.

Step 3:-

Select Nyquist plot as per function $G(s) H(s)$

Step 4:-

Analyse the sections as starting point and terminating point of plot.

Step 5:-

Mathematically find out ω_{pc} and intersection of Nyquist plot with Negative real axis by rationalizing $G(j\omega)\,H(j\omega)$

Step 6:-

With the knowledge of step 4 & 5, Sketch Nyquist plot.

Step 7:-

Count the number of encirclements N of $(-1+j0)$ by Nyquist plot. If this matches with criterion decided in step 2, system is stable, otherwise unstable.

1) A unity feedback control system has $G(S) = \frac{10}{S(S+1)(S+2)}$. Draw Nyquist plot and Comment on closed loop stability.

Solution

$$G(S)\,H(S) = \frac{10}{S(S+1)(S+2)}$$

$$H(S) = 1$$

Step 1 :

No. of poles in right half $P=0$.

No pole of $G(S)\,H(S)$ is right half

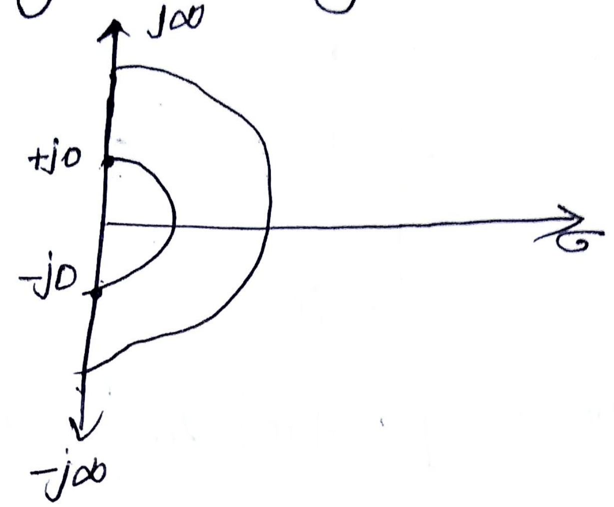
Step 2 :

for stability $N = -P = 0$

Nyquist plot should not encircle $(-1+j0)$ point for absolute stability of this system.

Step 3:-

As there is one pole at origin, it should be bypassed by semicircle.



Step 4:-

$$G(j\omega)\,H(j\omega) = \frac{10}{j\omega(1+j\omega)(2+j\omega)}$$

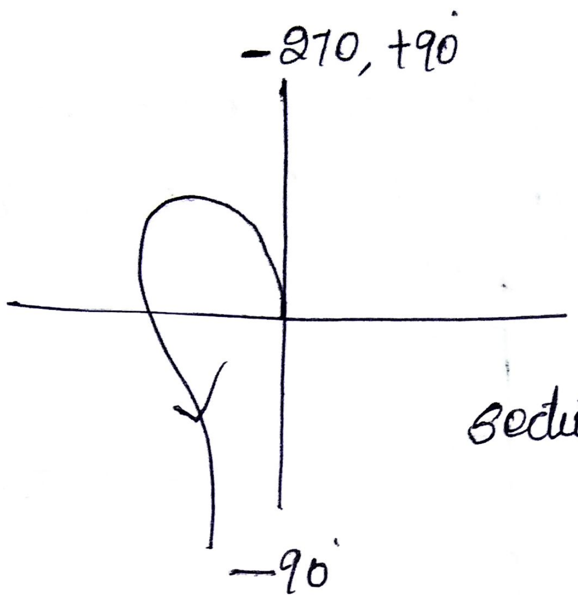
$$|G(j\omega)\,H(j\omega)| = \frac{10}{\omega\sqrt{1+\omega^2}\,\sqrt{4+\omega^2}}$$

$$\angle G(j\omega)\,H(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}\omega/2$$

Section: I $\boxed{s=+j\infty}$ to $\boxed{s=+j0}$ (ie) $\omega \to \infty$ to $\omega = 0$.

Starting point $\Rightarrow \omega \to \infty \Rightarrow 0\angle -270^\circ$

Terminating point $\Rightarrow \omega \to 0 \Rightarrow \infty \angle -90^\circ$

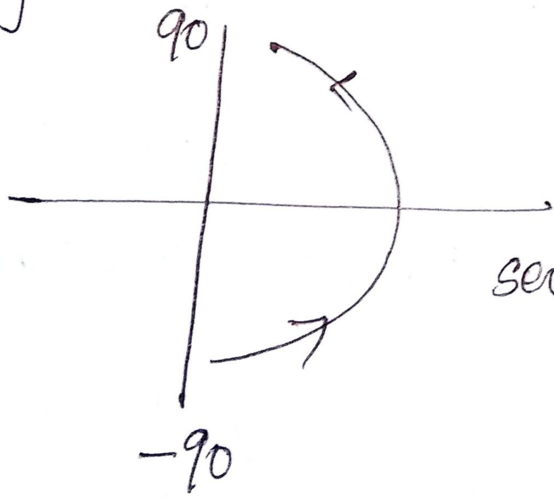


Section-I mapping

Section : II $\boxed{s=+j0}$ to $\boxed{s=-j0}$ (i.e) $\omega \to 0$ to $\omega \to -0$

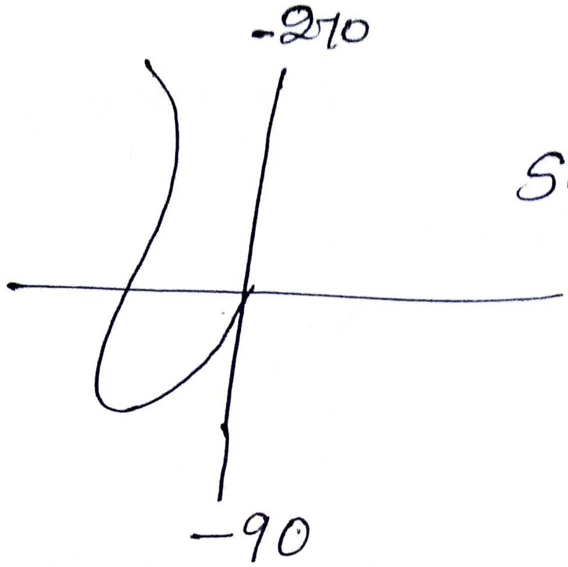
Starting point :- $\omega \to +0 \Rightarrow \infty \angle -90^\circ$

Terminating point :- $\omega \to -0 \Rightarrow \infty \angle +90^\circ$



Section II mapping

Section III: It is the Mirror image of section I about real axis



Section III mapping

Section IV :- It is the origin and not required to analysed.

Step 5: Find out intersection with negative real axis, Rationalise $G(j\omega)\,H(j\omega)$.

$$G(j\omega)\,H(j\omega) = \frac{10\,(-j\omega)\,(1-j\omega)\,(2-j\omega)}{(j\omega)\,(-j\omega)\,(1+j\omega)\,(1-j\omega)\,(2+j\omega)\,(2-j\omega)}$$

$$= \frac{-30\omega^2 - 10j\omega\,(2-\omega^2)}{\omega^2\,(1+\omega^2)\,(4+\omega^2)}$$

Separate real & Imag Part.

$$G(j\omega)\,H(j\omega) = \frac{-30\omega^2}{\omega^2(1+\omega^2)\,(4+\omega^2)} - \frac{10j\,\omega(2-\omega^2)}{\omega^2(1+\omega^2)\,(4+\omega^2)}$$

At $w = w_{pc}$ ⇒ imaginary part is zero.

$$10w(2-w^2) = 0$$

$10w = 0$ $\quad$ $2 - w^2 = 0$

$w = 0$ $\quad$ $w^2 = 2$

$w = \sqrt{2}$

Substitute $w_{pc} = \sqrt{2}$ is real part

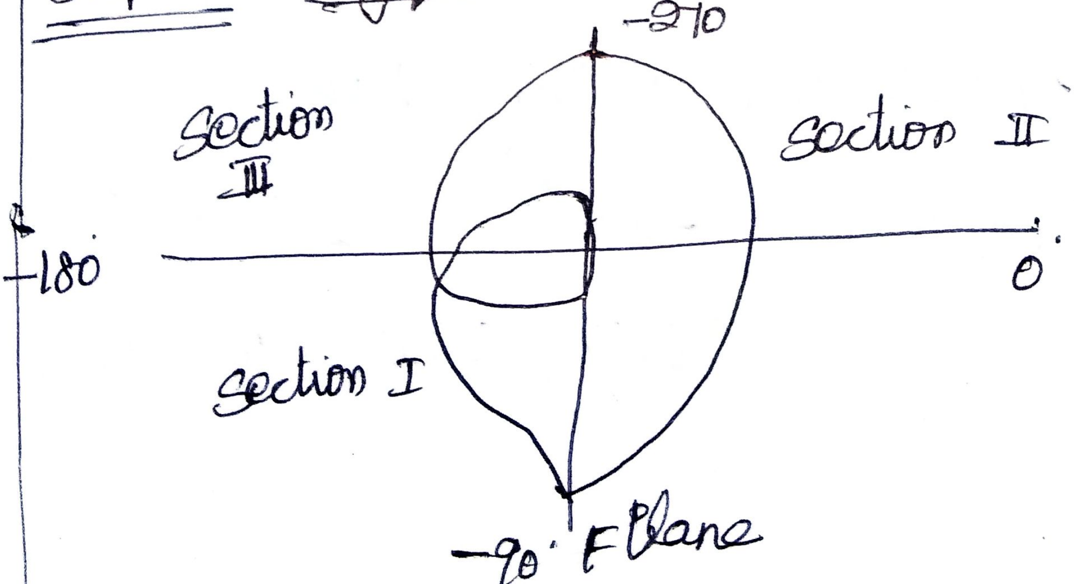
$$G(jw)\,H(jw)\Big|_{w=w_{pc}} = \frac{-30\times 2}{2(1+2)(4+2)} + j0$$

$$= -1.66 + j0.$$

$$\text{Gain Margin} = \frac{1}{|-1.66+j0|} = \frac{1}{1.66} = 0.6 = -4.43\,dB$$

As GM is negative, system is unstable because critical point is enclosed.

Step 6: Nyquist plot



N = +2

Step 7:

The Number of encirclement of $-1+j0$ are $\boxed{N=+2}$. as per step 2:- $\boxed{N=0}$. Hence it does not match. given system is unstable.

According to Mapping Theorem

$N=Z-P$

$2=Z-0$

$\boxed{Z=2}$

Actually there are 2 zeros of $1+G(s)H(s)$ encircled by Nyquist path.

Design of Componsators using Bode plot

It has often been observed that the performance of a Control system does not satisfy the given Specifications interms of accuracy, stability, damping, speed response and so on.

After design and testing if the system does not perform satisfactorily some changes may need to be introduced to achieve the desired results.

The changes could be in form of adjustment of forward path gain or insertion of componsating device in the Control system.

Componsators are required in following two cases namely.

(i) System is unstable :- Compensation is required to Stabilize it and also achieve the desired performance Specifications.

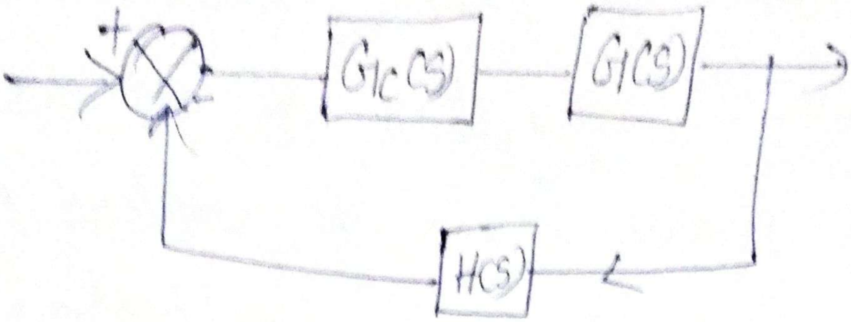
(ii) Compensation is required to achieve the improved performance Specifications

Compensation by Inserting N/w.

- Series Compensation (or) Cascade Compensation
  - Cascade lag N/w (phase lag occurs in f_L)
  - Cascade lead N/w (phase lead occurs in f_H)
  - Cascade lag-lead N/w
- Parallel Compensation (or) feedback Compensation
- Series-parallel Compensation

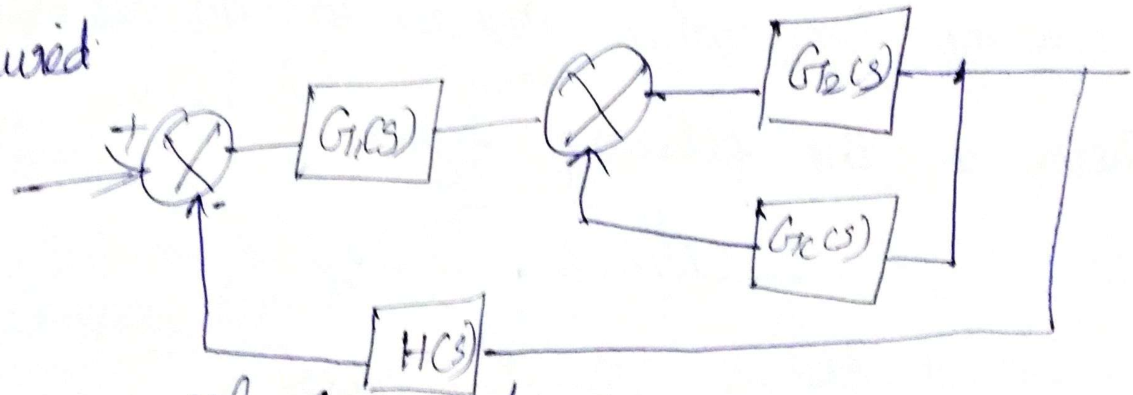
Cascade Compensation

If the Compensator $G_c(s)$ is placed in Series with forward path transfer function of the plant, the scheme is called series or cascade compensation. This requires additional amplifiers to increase the gain and also provide necessary isolation

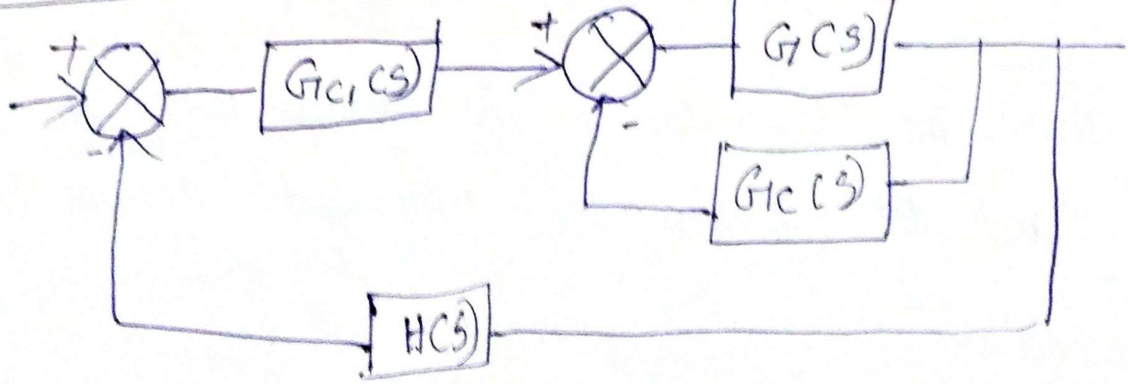


Feedback Compensation

If compensator $G_c(s)$ is placed in feedback path to provide an additional internal feedback loop. The energy transfer is from higher energy level towards lower energy level. Point Additional amplifiers are not required



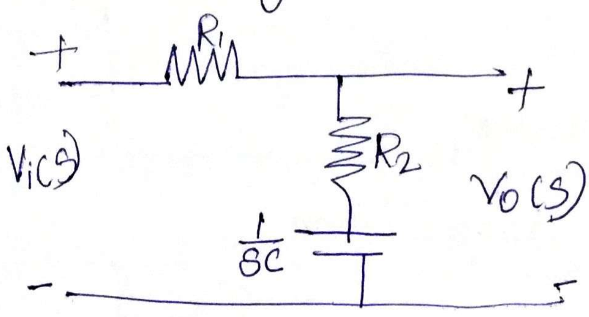
Series - parallel Compensation



The selection of proper Compensation Scheme depends on nature of signals availa[ble] in the system, Power levels at various points, available Components, economic Considerations, and designer's experience.

Lag Compensator

The Lag Compensator is an electrical network which produces a sinusoidal output having the phase lag when a sinusoidal input is applied. The lag Compensator circuit in the s domain is shown in the following figure.



Here the Capacitor is in series with the resistor R_2 and the output is measured across this Combination

$$T.F = \frac{V_o(s)}{V_i(s)} = \frac{1}{\alpha}\left(\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}\right)$$

$$\tau = R_2 C$$

$$\alpha = \frac{R_1+R_2}{R_2}$$

$s = j\omega$

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{\alpha}\left(\frac{j\omega + 1/\tau}{j\omega + 1/\alpha\tau}\right)$$

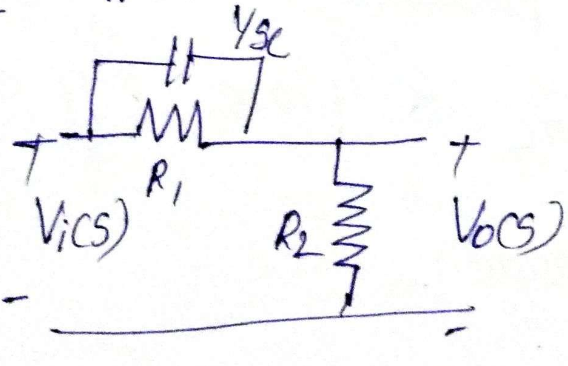
Phase angle $\phi = \tan^{-1}\omega\tau - \tan^{-1}\alpha\omega\tau$

In order to produce the phase lag at the output of this Compensator. The phase angle of the transfer function should be negative. This will happen when

$\alpha > 1$

Lead Compensator

The lead compensator is an electrical Network which produces a sinusoidal output having phase lead when sinusoidal input is applied.



$$T.F = \frac{V_o(s)}{V_i(s)} = \beta\left(\frac{s\tau+1}{\beta s\tau+1}\right)$$

$$\tau = R_1 C$$

$$\beta = \frac{R_2}{R_1+R_2}$$

$s = j\omega$

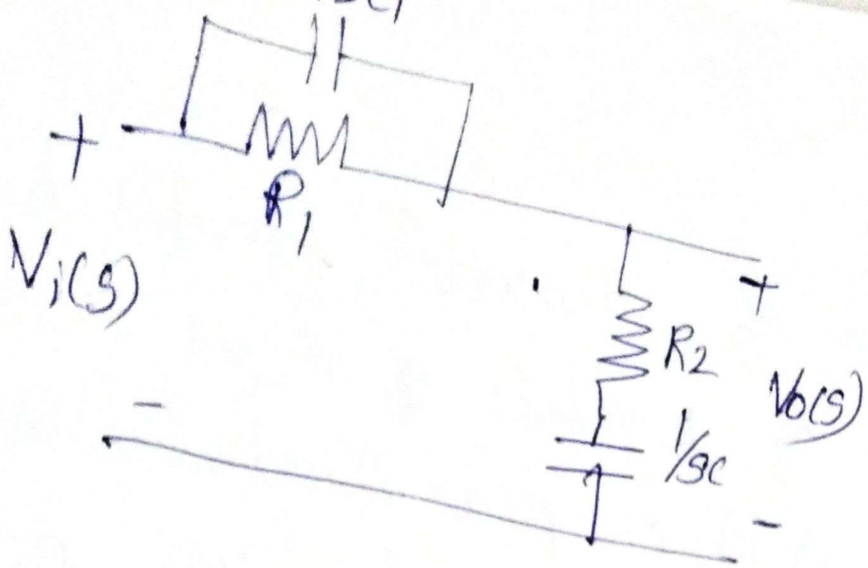
$$\frac{V_o(j\omega)}{V_i(j\omega)} = \beta\left(\frac{j\omega\tau+1}{\beta j\omega\tau+1}\right)$$

$$\phi = \tan^{-1}\omega\tau - \tan^{-1}\beta\omega\tau$$

In order to produce the phase lead at the output of this compensator, the phase angle of the T.F should be positive. This will happen when $0<\beta<1$. Therefore, zero will be nearer to origin in pole-zero configuration of the lead compensator.

Lag Lead Compensator

Lag Lead compensator is an electrical network which produces phase lag at one frequency region & phase lag at other frequency region.



$$T.F = \frac{V_o(S)}{V_i(S)} = \beta\left(\frac{S\tau_1+1}{\beta S\tau_1+1}\right)\frac{1}{\alpha}\left(\frac{S+\frac{1}{\tau_2}}{S+\frac{1}{\alpha\tau_2}}\right)$$

$\alpha\cdot\beta = 1$

$$\frac{V_o(S)}{V_i(S)} = \left(\frac{S+\frac{1}{\tau}}{S+\frac{1}{\beta\tau_1}}\right)\left(\frac{S+\frac{1}{\tau_2}}{S+\frac{1}{\alpha\tau_2}}\right)$$

$\tau_1 = R_1 C_1$

$\tau_2 = R_2 C_2$

Unit IV - Concepts of Stability Analysis

Concept of stability - Bounded - Input Bounded - Output Stability. Routh stability Criterion - Relativ stability. Root locus Concept - Guidelines for Sketching root locus - Nyquist stability Criterion

Stability:-

The term stability refers to the stable working condition of control system. Every working system is designed to be stable. In a stable system the response or output is stable, finite and predictable.

1] A system is stable if its output is bounded (finite) for any bounded input

2] A system is asymptotically stable if the absence of the input, the output tends towards zero.

3) A system is stable if for a bounded disturbing input signal, the output vanishes

ultimately as t approaches infinity.

4) A system is unstable if for bounded disturbing input signal the output is of infinite amplitude.

5) For a bounded input signal, if the output has constant amplitude oscillations then the system may be stable or unstable under some limited constraints. Such a system is called limitedly stable.

6) If a system output is stable for all variations of its parameters then the system is called absolutely stable system.

7) If a system output is stable for a limited range of variations of its parameters, then the system is called conditionally stable system.

Bounded - Input Bounded - Output (BIBO)

Stability

A linear relaxed system is said have BIBO stability if every bounded (finite) input results in a bounded (finite) output.

A test for BIBO stability can be obtained from convolution theorem.

The response in time domain

$$c(t) = \int_0^\infty m(\tau)\, r(t-\tau)\, d\tau$$

Stability of the system depending on the location of roots of characteristic equation

Stable system → All roots of characteristic Equation have negative real parts

Unstable system → Any roots of CE has positive real parts

(or)

CE has repeated roots on the imaginary axis unstable → One or more non repeated roots of the CE are lying on the imaginary axis

(or)

CE has single root at origin

(or)

CE has repeated roots at origin.

Limitedly (or) Marginally stable ⇒ In system with one or more repeated roots on imaginary axis or with single root at origin.

Coefficients of characteristic polynomial

Coefficients ⇒ positive ⇒ No zeros

[All roots are in left half of s plane]

Coefficients $\Rightarrow 0 \Rightarrow$ Roots may be imaginary axis or right half of s plane.

Coefficients $\Rightarrow$ -ve $\Rightarrow$ atleast one root in right half of s plane.

Routh HURWITZ Criterion

It is an analytical procedure for determining whether all the roots of a polynomial have negative real part or not.

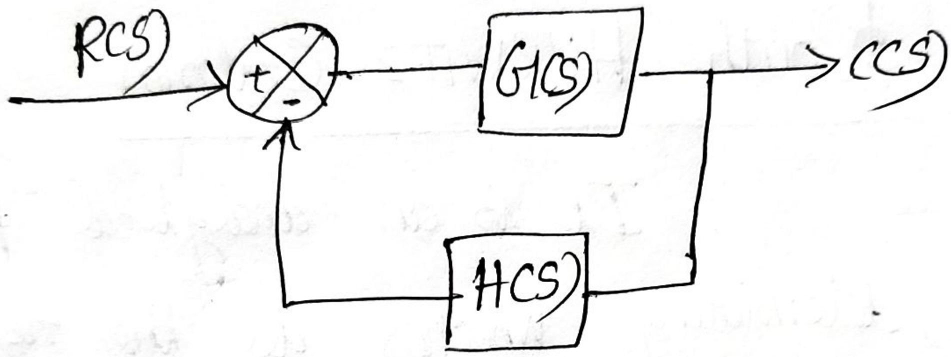
Types

Case I: Normal Routh array [Non zero elements in the first column of routh array]

Case II: A row of all zeros

Case III :- First element of a row is zero. But some or other elements are not zero.

Formation of Routh Array



$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)\,H(S)}$$

$CE = 1 + G(S)\,H(S) = 0 \Rightarrow$ characteristic Equation

① Using Routh Criterion, determine the stability of the system represented by the C.E, $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic Equation.

Solution

The given C.E is 4^{th} order equation so it has 4 roots.

highest power of s is even Number.

first row → Coefficients of even powers of s

second row → Coefficients of odd power of s

s^4 :	1	18	5	Row 1
s^3 :	8	16		Row 2

The elements of s^3 row can be divided by 8 to simplify the computations

s^4 :	1	18	5	Row1
s^3 :	1	2		Row 2
s^2 :	16	5		Row 3
s^1 :	1.7			Row 4
s^0 :	5			Row 5

$$s^2 \Rightarrow \frac{1\times18 - 2\times1}{1} \qquad \frac{5\times1 - 0\times1}{1}$$

$$s^2 = 16 \qquad 5$$

$$s^0 = \frac{1.7\times5 - 16\times0}{1.7}$$

$$s^0 = 5$$

$$s^1 = \frac{16\times2 - 5\times1}{16}$$

$$= 1.7$$

All elements in 1st column is positive

Result

1) The system is stable.

2) All 4 roots are lying on the left half of s plane.

2) Construct Routh array and determine the stability of the system whose characteristic equation is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$. Also determine the Number of roots lying on right half of s-plane, left half of s-plane and on imaginary axis.

<u>Solution:</u>

The characteristic equation of the system is $s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

s^6 :	1	8	20	16	Row 1
s^5 :	2	12	16		Row 2

s^6 : 1 8 20 16 .... Row1

s^5 : 1 6 8 ...... Row2

s^4 : 1 6 8 .... Row 3

s^3 : 0 0 .... Row 3

s^3 : 1 3 .... Row 4

s^2 : 3 8 .... Row 5.

s^1 : 0.33 .... Row 6

s^0 : 8 .... Row 7

$$s^4 \rightarrow \frac{1\times 8 - 6\times 1}{1} \qquad \frac{1\times 20 - 8\times 1}{1} \qquad \frac{1\times 16 - 0\times 1}{1}$$

$s^4 \rightarrow$ 2 12 16

divided by 2

$s^4 \rightarrow$ 1 6 8

$s^3 \rightarrow \dfrac{1\times 6 - 6\times 1}{1}$ $\qquad$ $\dfrac{1\times 8 - 8\times 1}{1}$

$s^3 \rightarrow 0 \qquad 0$

The auxiliary equation is $A = s^4 + 6s^2 + 8$. On differentiating A with respect to s we get

$$\frac{dA}{ds} = 4s^3 + 12s$$

$s^3 \rightarrow 4 \quad 12$

divided by 4

$s^3 \rightarrow 1 \quad 3$

$s^2 \rightarrow \dfrac{1\times 6 - 3\times 1}{1}$ $\qquad$ $\dfrac{1\times 8 - 0\times 1}{1}$

$s^2 \rightarrow 3 \qquad 8$

$s^1 \rightarrow \dfrac{3\times 3 - 8\times 1}{3}$

$s^1 \rightarrow 0.33$

$s^0 \rightarrow \dfrac{0.33\times 8 - 0\times 3}{0.33}$

$s^0 \rightarrow 8$

On examining the elements of 1st column of routh array it is observed that there is no sign change. The row with all zeros indicate the possibility of roots in imaginary axis.

Hence the system is limitedly or marginally stable.

The auxiliary polynomial is

$$s^4 + 6s^2 + 8 = 0.$$

Let $s^2 = x$.

$$\therefore x^2 + 6x + 8 = 0.$$

The roots of Quadratic are

$$x = \frac{-6 \pm \sqrt{6^2 - 4(8)}}{2}$$

$$= -3 \pm 1 = -2 \text{ (or) } 4$$

$$s = \pm\sqrt{x} = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$s = +j\sqrt{2}, -j\sqrt{2}, +j2 \text{ and } -j2$$

The roots of auxiliary polynomial are also roots of CE. Hence 4 roots are lying on imaginary axis and the remaining 2 roots are lying on the left half of s plane.

Result

1. The system is limitedly or Marginally stable

2. Four roots are lying on imaginary axis and the remaining two roots are lying on the left half of splane.

(3) Construct Routh array and determine the stability of the system represented by characteristic equation $s^5+s^4+2s^3+2s^2+3s+5=0$. Comment on the location of the roots of characteristic equation

Solution:

s^5 : 1 2 3

s^4 : 1 2 5

s^3 : ε -2

s^2 : $\frac{2\varepsilon+2}{\varepsilon}$ 5

s^1 : $\frac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2}$

s^0 : 5

$s^3 \Rightarrow \frac{1\times2-2\times1}{1}$ $\frac{1\times3-5\times1}{1}$

$s^3 \Rightarrow 0 \quad -2$

Replaced 0 by ε

$s^3 = \varepsilon \quad -2$

On letting $\varepsilon \to 0$ we get

s^5 : 1 2 3

s^4 : 1 2 5

s^3 : 0 -2

s^2 : ∞ 5

s^1 : -2

s^0 : 5

It is found that there is two sign changes

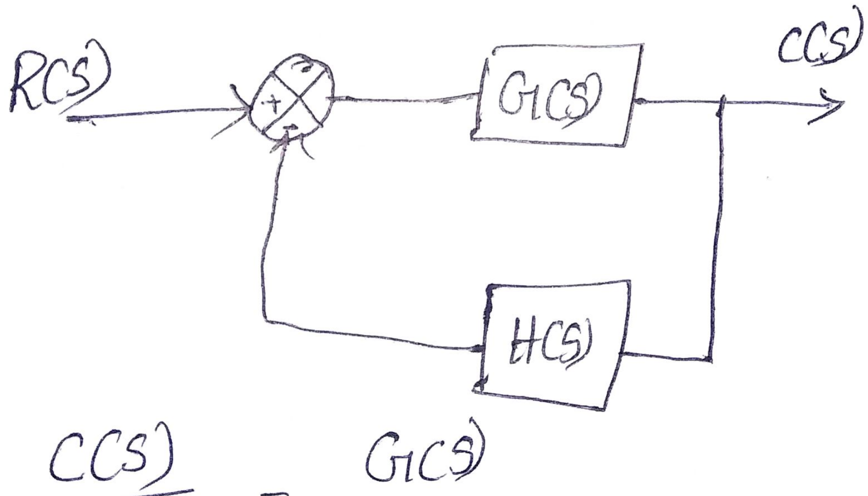
Result

1) The system is unstable.
2) The roots are lying on the right half of s plane & 3 roots are lying on the left half of s plane.

ROOT LOCUS

The roots of characteristic equation is a function of open loop gain K. When the gain is varied from 0 to ∞, the roots of characteristic equation will take different values. When K=0, the roots are given by open loop poles. When K→∞, the roots will takes the value of open loop zeros.

The path taken by the roots of characteristic equation when open loop gain K varied from 0 to ∞ are called Root locus



$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

C.E is $1+G(S)H(S)=0$

$$G(S)H(S)=-1$$

Magnitude Criterion $|G(s)\ H(s)| = 1$

Angle criterion $\angle G(s)\ H(s) = \pm 180°(2q+1)$

Where $q = 0, 1, 2, 3 \ldots$

Procedure

Step 1 :- Locate the pole and zeros for the given T.F

Step 2 :- Determine the root locus on real axis

Step 3 :- Calculate the angle of asymptotes and centroid.

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$\angle \text{Asymptotes} = \frac{\pm 180°(2q+1)}{n-m}$$

$$\frac{dk}{ds} = 0.$$

Step 4: Find the breakaway & break in points

$$\frac{dk}{ds} = 0$$

Step 5: Find the angle of Departure & angle of arrival.

Angle of Departure : (Breakaway point)

If there is complex pole ⇒ Angle of Departure find.

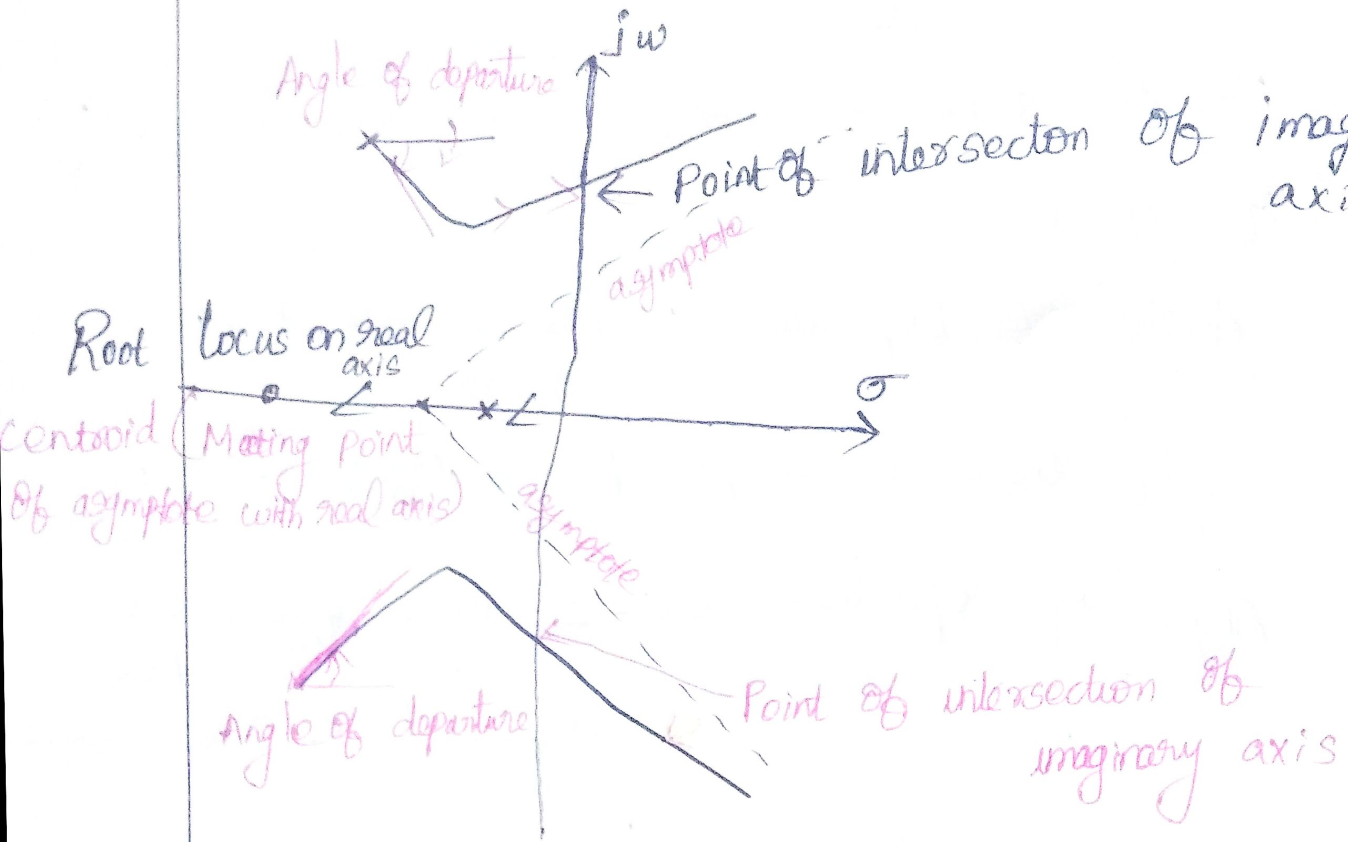
Angle of departure $= 180° - \Sigma$ angle from other poles $+ \Sigma$ angle from other zeros

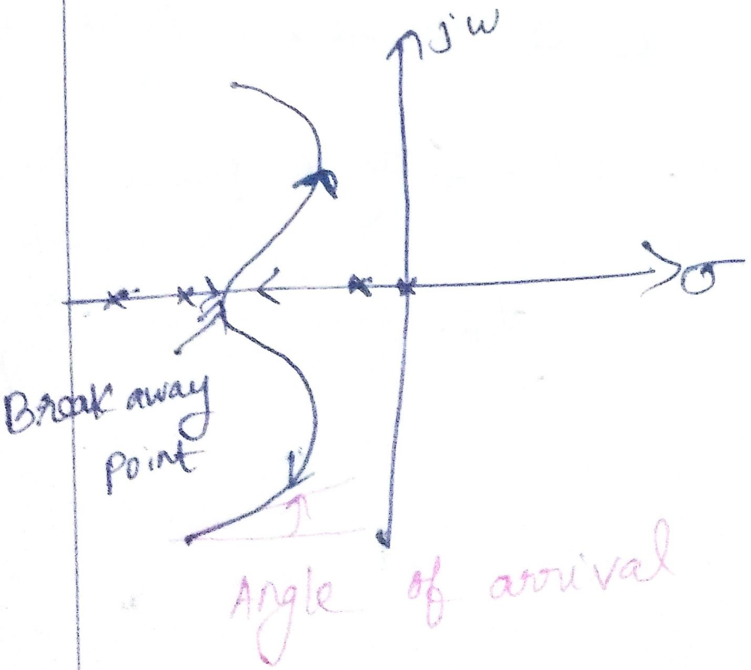
Angle of Arrival : (Break in point)

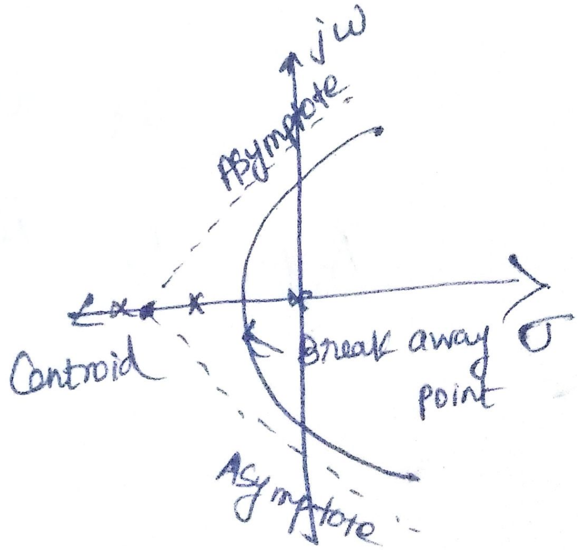
If there is complex zero ⇒ Angle of arrival

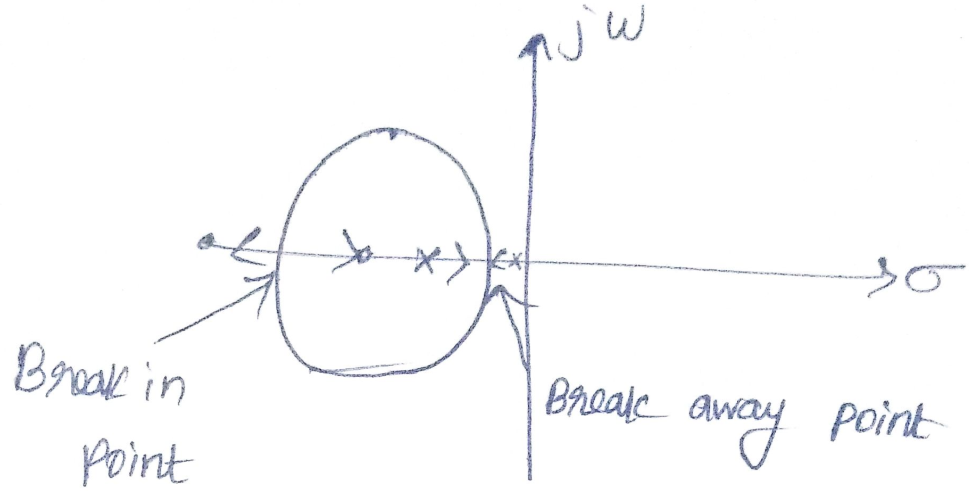
Angle of Arrival $= 180° - \Sigma$ angle from other zeros $+ \Sigma$ angle from other poles

Step 6: Find Crossing point of Imaginary axis.








jw
σ
Break in
Point
Break away point

Problem 1:

A unity feedback control system has an open loop T.F $G(s) = \dfrac{K \rightarrow \text{zeros} \rightarrow m}{s(s^2+4s+13) \rightarrow \text{poles} \rightarrow n}$. Sketch the root locus.

Sol:

Step 1: To locate poles and zeros.

Poles of OLTF are roots of eq. $s(s^2+4s+13)=0$

$$s = \frac{-4 \pm \sqrt{4^2 - 4\times 13}}{2} = -2 \pm j3$$

poles are $s = 0, -2+j3, -2-j3$

Denote poles $\Rightarrow P_1, P_2$ & P_3

$P_1 = 0, \quad P_2 = -2+j3, \quad P_3 = -2-j3$

Poles $\Rightarrow$ X.

Step 2: To find root locus on real axis

Test point between	No. of poles and zeros on right side of test point	Real axis between
0 to $-\infty$	odd no. of poles & zeros $\Rightarrow s = 0$ 1 pole.	0 to $-\infty$ (Root locus)

Step 3: To find angle of asymptotes and centroid:

No. of poles $= 3$ so $n = 3$.

$n - m = 3 - 0 = 3.$

$$\text{Angle of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m} \qquad q = 0, 1, \ldots \frac{n-m}{3.}$$

$q = 0, 1, 2, 3.$

$n = 3, \; m = 0, \; q = 0, 1, 2, 3.$

when $q = 0$, Angles $= \dfrac{\pm 180}{3} = \pm 60^\circ$

$q = 1$, Angles $= \pm \dfrac{180 (\cancel{3})}{\cancel{3}} = \pm 180^\circ$

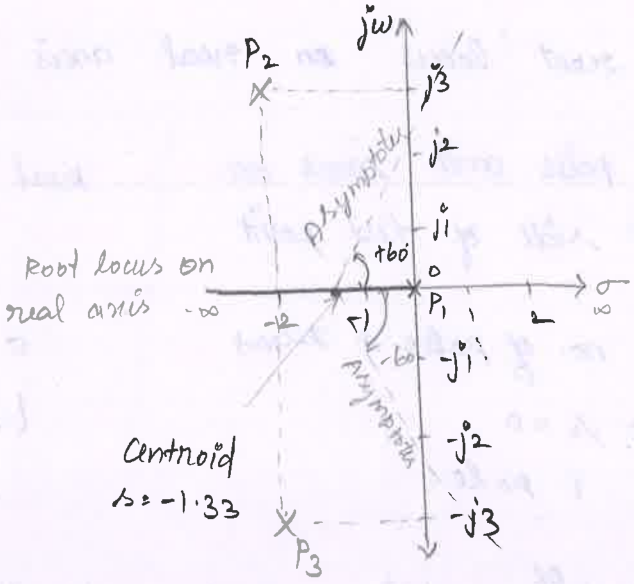
$q = 2$, Angles $= \pm \dfrac{\overset{60}{\cancel{180}} (5)}{\cancel{3}} = \pm 300 = \pm 60^\circ$

$q = 3$, Angles $= \pm \dfrac{\overset{60}{\cancel{180}} (7)}{\cancel{3}} = \pm 420 = \pm 60^\circ$

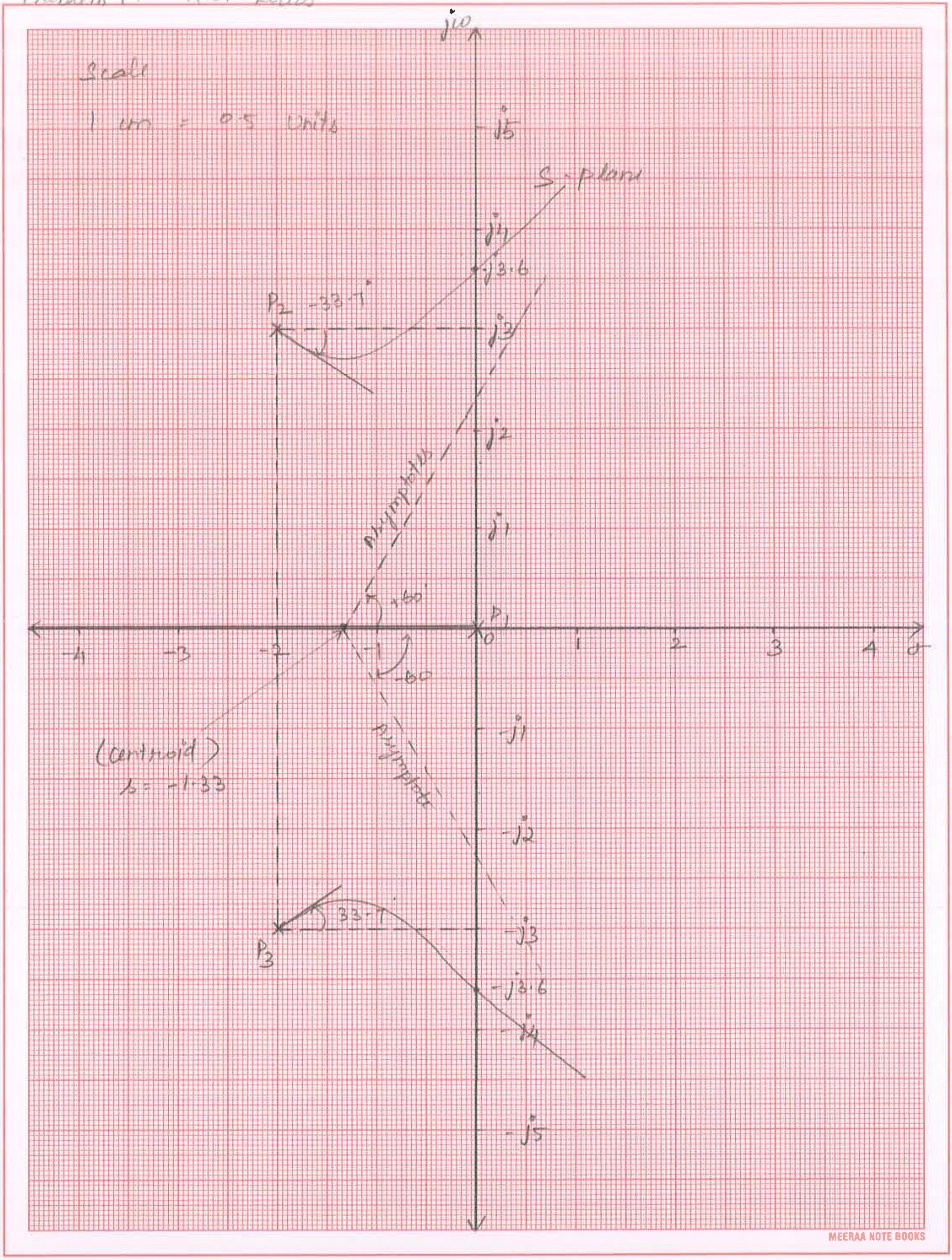
$$\text{centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0 - 2 + \cancel{j\sqrt{3}} - 2 - \cancel{j\sqrt{3}} - 0}{3}$$

$$= -\frac{4}{3} = -1.33$$



Problem 1: Root Locus



Step 4: To find the breakaway and breakin points.

$$CLTF \Rightarrow \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{K/s(s^2+4s+13)}{1+K/s(s^2+4s+13)}$$

$$= \frac{K}{s(s^2+4s+13)+K} \longrightarrow C.E$$

C.E : $s(s^2+4s+13)+K = 0$

$$s^3+4s^2+13s+K = 0$$

$$K = -(s^3+4s^2+13s)$$

Diff above eq. with respect to s, we get,

$$\frac{dK}{ds} = -(3s^2+8s+13)$$

Put $\frac{dK}{ds} = 0$

$$-(3s^2+8s+13) = 0 \Rightarrow 3s^2+8s+13 = 0$$

$$s = \frac{-8 \pm \sqrt{8^2-4\times13\times3}}{2\times3} = -1.33 \pm j1.6$$

$$s_1 = -1.33 + j1.6$$

$$s_2 = -1.33 - j1.6$$

sub s_1 in K eq.

$$K = -(s^3+4s^2+13s)$$

$$= -[(-1.33+j1.6)^3 + 4(-1.33+j1.6)^2 + 13(-1.33+j1.6)]$$

$$= 12.59 - j8.17$$

Sub s_2 in K eq.

$$K = -\left[(-1.33 - j1.6)^3 + 4(-1.33 - j1.6)^2 + 13(-1.33 \mp j1.6)\right]$$

$$= 12.59 \mp j8.17$$

K is not +ve and real.

Hence there is neither break away point nor break in point.

Step 5 : To find angle of departure.

Let us consider pole P_2.

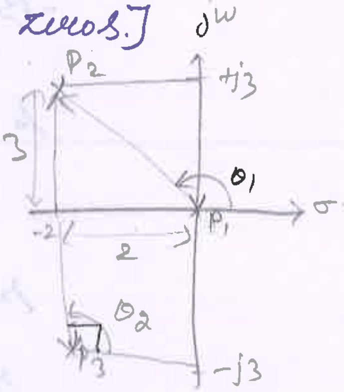
$$\theta_1 = 180^\circ - \tan^{-1}\left(\frac{3}{2}\right)$$

$$= 123.7^\circ$$

$$\theta_2 = 90^\circ$$

Angle of departure from complex pole P_2 } = 180° − [Sum of angle of complex pole P_2 from other poles]

+ [Sum of angle of complex pole P_2 from zeros.]



$$= 180 - (\theta_1 + \theta_2)$$

$$= 180 - (123.7 + 90^\circ)$$

$$= -33.7^\circ$$

Angle of departure from Pole P_3 } = − (Angle of departure from P_2)

$$= 33.7^\circ$$

Mark AOD at Complex pole using protector.

Step 6 : To find the crossing point on imaginary axis

$C.E \Rightarrow s^3 + 4s^2 + 13s + K = 0$

Put $s = j\omega$

$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0$

$-j\omega^3 - 4\omega^2 + 13j\omega + K = 0$

1st Find

Equating imaginary part

$13\omega - \omega^3 = 0$

$-\omega^3 = -13\omega$

$\omega^2 = 13$

$\omega = \pm\sqrt{13}$

$\boxed{\omega = \pm 3.6}$

After

Equating Real part to 0

$-4\omega^2 + K = 0$

$4\omega^2 = K$

$4(13) = K$

$\boxed{K = 52}$

Crossing point of root locus $= \pm j3.6$

Value of K at this crossing point, $K = 52$.

This is limiting value of K for the stability of system.

Problem 2 :

Sketch the root locus of the system whose open loop transfer function is $G(s) = \dfrac{K}{s(s+2)(s+4)}$. Find the value of K so that the damping ratio of the closed loop system is 0.5.

Sol :

Step 1 : To locate poles and zeros :

poles $\Rightarrow$ OLTF $\Rightarrow$ $s(s+2)(s+4) = 0$.

poles are lying at, $s = 0, -2, -4$.

poles as P_1, P_2 and P_3

$P_1 = 0$, $P_2 = -2$, $P_3 = -4$

poles marked by x.

<u>Step 2</u>: To find the root locus on real axis.

Test point between	No. of poles and zeros on real axis between right side of test point	
0 to −2	odd no. = 1 poles ($s = 0$)	0 to −2 Root locus
−2 to −4	even no = 2 poles ($s = 0$) ($s = -2$)	−2 and −4 Not root locus
−4 to −∞	odd no = 3 poles ($s = 0$, $s = -2$, $s = -4$)	−4 to −∞ Root locus.

<u>Step 3</u>: Find angle of asymptotes and centroid

Asymptotes $= n - m = 3 - 0 = 3$.

$$\text{Angle of asymptotes} = \frac{\pm 180 (2q + 1)}{n - m}$$

$n = 3$

$m = 0$

$q = 0, 1, 2, 3$

At $q = 0$, Angle of asymptotes $= \frac{\pm 180(1)}{3}$

$= \pm 60$

At $q = 1$ Angle of asymptotes $= \frac{\pm 180^\circ (3)}{3} = \pm 180^\circ$

At $q = 2$ Angle of asymptotes $= \frac{\pm 180(5)}{3} = \pm 300^\circ$

At $q = 3$ Angle of asymptotes $= \frac{\pm 180^\circ (7)}{3} = \pm 420^\circ$

$$\text{centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

$$= \frac{0-2-4-0}{3} = -2$$

$$\boxed{\text{centroid} = -2}$$

Step 4: To find breakaway and breakin points

Closed loop Transfer Function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)} = \frac{K/s(s+2)(s+4)}{1+K/s(s+2)(s+4)}$$

$$= \frac{K}{s(s+2)(s+4)+K}$$

C.E $\quad s(s+2)(s+4)+K=0$

$$s^3+6s^2+8s+K=0$$

$$K = -(s^3+6s^2+8s) \longrightarrow ①$$

$$\frac{dK}{ds} = -(3s^2+12s+8)$$

Sub $\frac{dK}{ds}=0$; $\quad 3s^2+12s+8=0$

$$s=-0.845,\ s=-3.15$$

sub $s=-0.845$ ①

$$K = -[(-0.845)^3+6(-0.845)^2+8(-0.845)]$$

$$= -(-0.603+4.284-6.76)$$

$\boxed{K = 3.08}$ => Real and Positive.

Sub $s = -3.15$ in ①

$K = -[(-3.15)^3 + 6(-3.15)^2 + 8(-3.15)]$

$\boxed{K = -3.08}$ => Not real and +ve.

Step 5 : Angle of arrival and angle of departure.

C.E => $s^3 + 6s^2 + 8s + K = 0$

sub $s = j\omega$

$(j\omega)^3 + 6(j\omega)^2 + 8j\omega + K = 0$

=> $-j\omega^3 - 6\omega^2 + j8\omega + K = 0$

$(K - 6\omega^2) + j(8\omega - \omega^3) = 0$

Equating imaginary part

$-j\omega^3 + j8\omega = 0$

$j\omega^3 = -j8\omega$

$\omega^2 = 8$

$\omega = \pm\sqrt{8}$

$\boxed{\omega = \pm 2.8}$

Equating real part.

$-6\omega^2 + K = 0$

$K = 6\omega^2$

$K = 6 \times 8$

$\boxed{K = 48}$

Crossing point of root locus is $\pm j2.8$

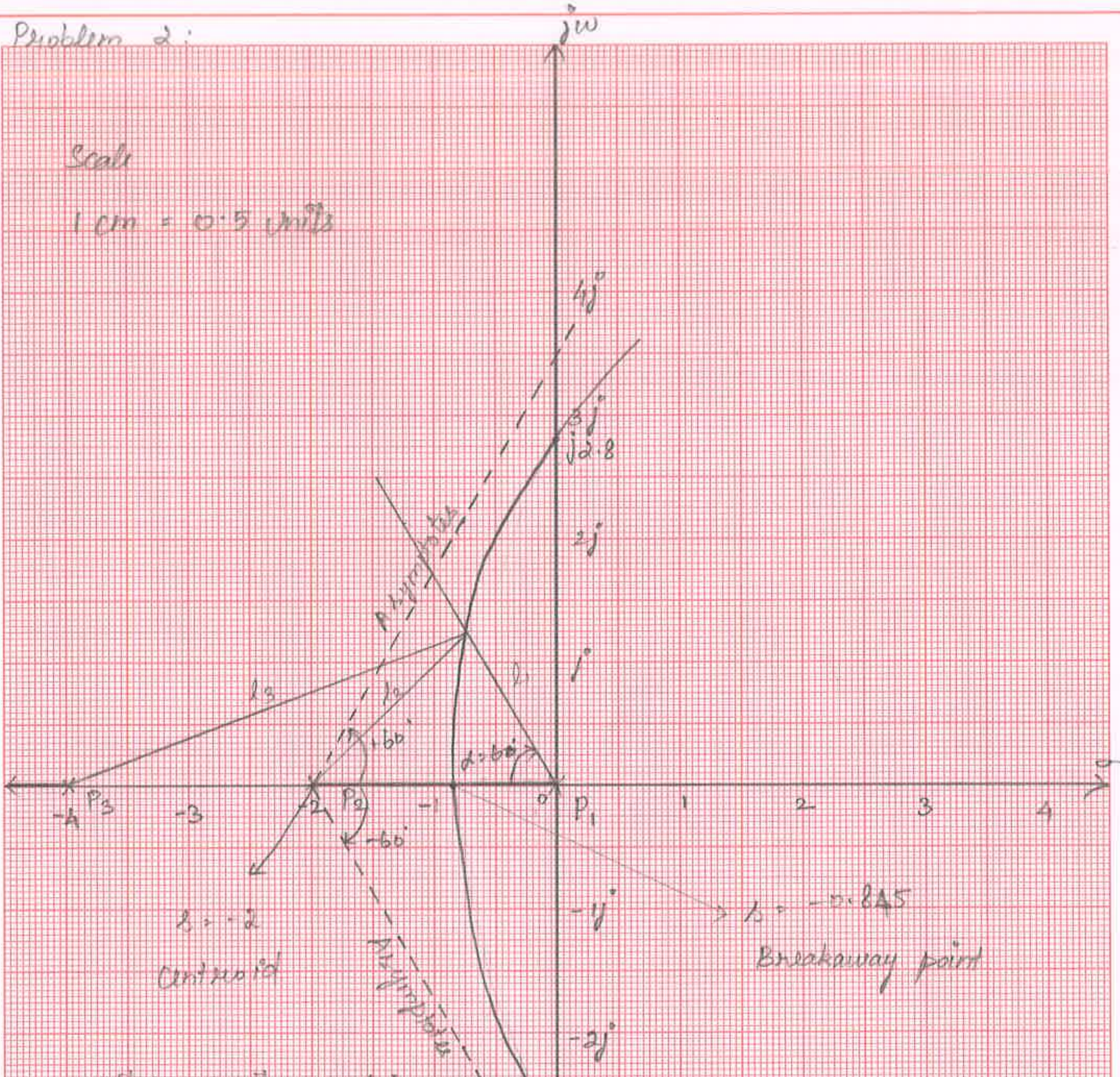
$K = 48$.

To find value of K corresponding to $\zeta = 0.5$

$\zeta = 0.5$.

$\alpha = \cos^{-1}\zeta = \cos^{-1} 0.5 = 60^\circ$

Problem 2:



$\alpha = \cos^{-1}\zeta = \cos^{-1} 0.5 = 60^\circ$

$l_1 = 2.6\ cm = 2.6 \times 0.5 = 1.3$ Units

$l_2 = 3.5\ cm = 3.5 \times 0.5 = 1.75$ Units

$l_3 = 7\ cm = 7 \times 0.5 = 3.5$ Units

$K = \dfrac{l_1 \times l_2 \times l_3}{1}$

$= 1.3 \times 1.75 \times 3.5$

$= 7.96 \approx 8$

Draw angle between OP and negative real axis is $60°$ $(\alpha = 60°)$ $\Rightarrow$ Dominant pole $\Rightarrow S_d$.

$$K_{sd} = \frac{\text{Product of length of Vector from all poles to the point, } s = s_d}{\text{Product of length of Vector from all zeros to the point, } s = s_d}$$

$$= \frac{l_1 \times l_2 \times l_3}{1}$$

$$= \frac{1.3 \times 1.75 \times 3.5}{1}$$

$$= 7.96 \approx 8.$$

Problem 3:

Sketch the root locus for the unity feedback system whose OLTF $G(s) = \dfrac{K}{s(s+4)(s^2+4s+20)}$ (AU/J -2014)

Sol:

Step 1: To locate poles and zeros

$$s(s+4)(s^2+4s+20) = 0$$

$$s = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 20}}{2} = -2 \pm j4$$

poles $\Rightarrow$ $s = 0$

$s = -4$

$s = -2 + j4$

$s = -2 - j4$

Zeros are lying at infinity.

poles $\Rightarrow$ P_1 P_2 P_3 P_4.

Where $P_1 = 0$

$P_2 = -4$

$P_3 = -2 + j4$

$P_4 = -2 - j4$

Step 2 : To find root locus on real axis.

$s = 0$ $\quad$ $s = 0$ to -4 $\quad$ Root locus.

odd (1)

$s = -4$, $s = -4$ to $-\infty$ $\quad$ Not root locus

Total $\Rightarrow$ (2)

even.

Step 3 : To find asymptotes and centroid.

$n = 4, \quad m = 0.$

$n - m = 4.$

$q = 0, 1, \dots n-m \Rightarrow q = 0, 1, 2, 3, 4$

At $q = 0 \Rightarrow$ Angles $= \pm \dfrac{180\,(2(0)+1)}{4}$

$= \pm \dfrac{180}{4}$

$q = 1. \Rightarrow$ Angles $= \pm \dfrac{180\,(2+1)}{4} = \dfrac{540}{4} = 135°$

$q = 2 \Rightarrow$ Angles $= \pm \dfrac{(180\,(5))}{4} \Rightarrow 225 = 45$

$q = 3 \Rightarrow$ Angles $= \pm \dfrac{180\,(7)}{4} \Rightarrow 315$

$$\text{centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

$$= \frac{0-4-2+j4-2-j4-0}{4-0} = \frac{-8}{4} = -2$$

Step 4: To find the breakaway and breakin point.

$$\text{CLTF} \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{K/s(s+4)(s^2+4s+20)}{1+K/s(s+4)(s^2+4s+20)}$$

$$= \frac{K}{s(s+4)(s^2+4s+20)+K}$$

C.E

$$\text{C.E} \Rightarrow s(s+4)(s^2+4s+20)+K = 0$$

$$K = -[(s^2+4s)(s^2+4s+20)] \rightarrow (1)$$

$$= -[s^4+4s^3+20s^2+4s^3+16s^2+80s]$$

$$K = -[s^4+8s^3+36s^2+80s]$$

$$\frac{dK}{ds} = -[4s^3+24s^2+72s+80]$$

By Lin's method or by calc.

$s_1 = -2$

$s_2 = -2+2.45j$

$s_3 = -2-2.45j$

For check. Sub s value in ①

$s = -2 \Rightarrow k = -[s^4 + 8s^3 + 36s^2 + 80s]$

$= -[(-2)^4 + 8(-2)^3 + 36(-2)^2 + 80(-2)]$

$= 64.$

$s = -2-2j \Rightarrow k = -[(-2-2j)^3(-2-2j) + 8(-2-2j)^3 + 36(-2-2j)^2 + 80(-2-2j)]$

$= 99.99 \approx 100$

$s = -2+2j \Rightarrow k = -[(-2+2j)^3(-2+2j) + 8(-2+2j)^3 + 36(-2+2j)^2 + 80(-2+2j)]$

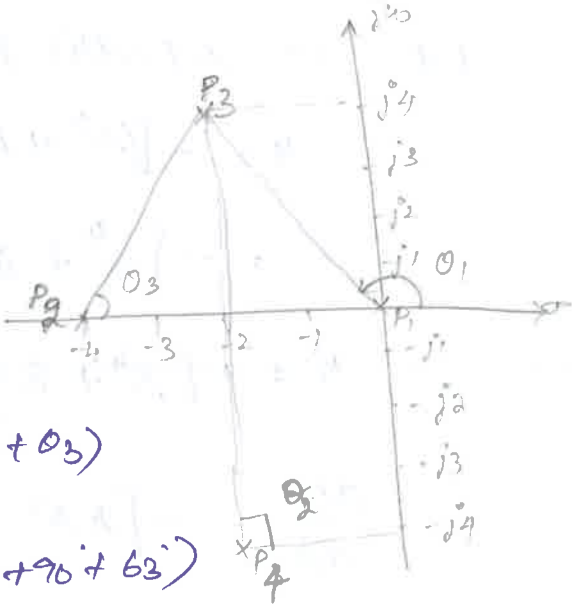
$= 100$

Step 5 : To find angle of departure :

$\theta_1 = 180^\circ - \tan^{-1}\frac{4}{2} = 117.$

$\theta_2 = 90^\circ$

$\theta_3 = \tan^{-1}\frac{4}{2} = 63$



Angle of departure from complex pole P_3 $\Big\}$ $= 180 - (\theta_1 + \theta_2 + \theta_3)$

$= 180^\circ - (117 + 90^\circ + 63^\circ)$

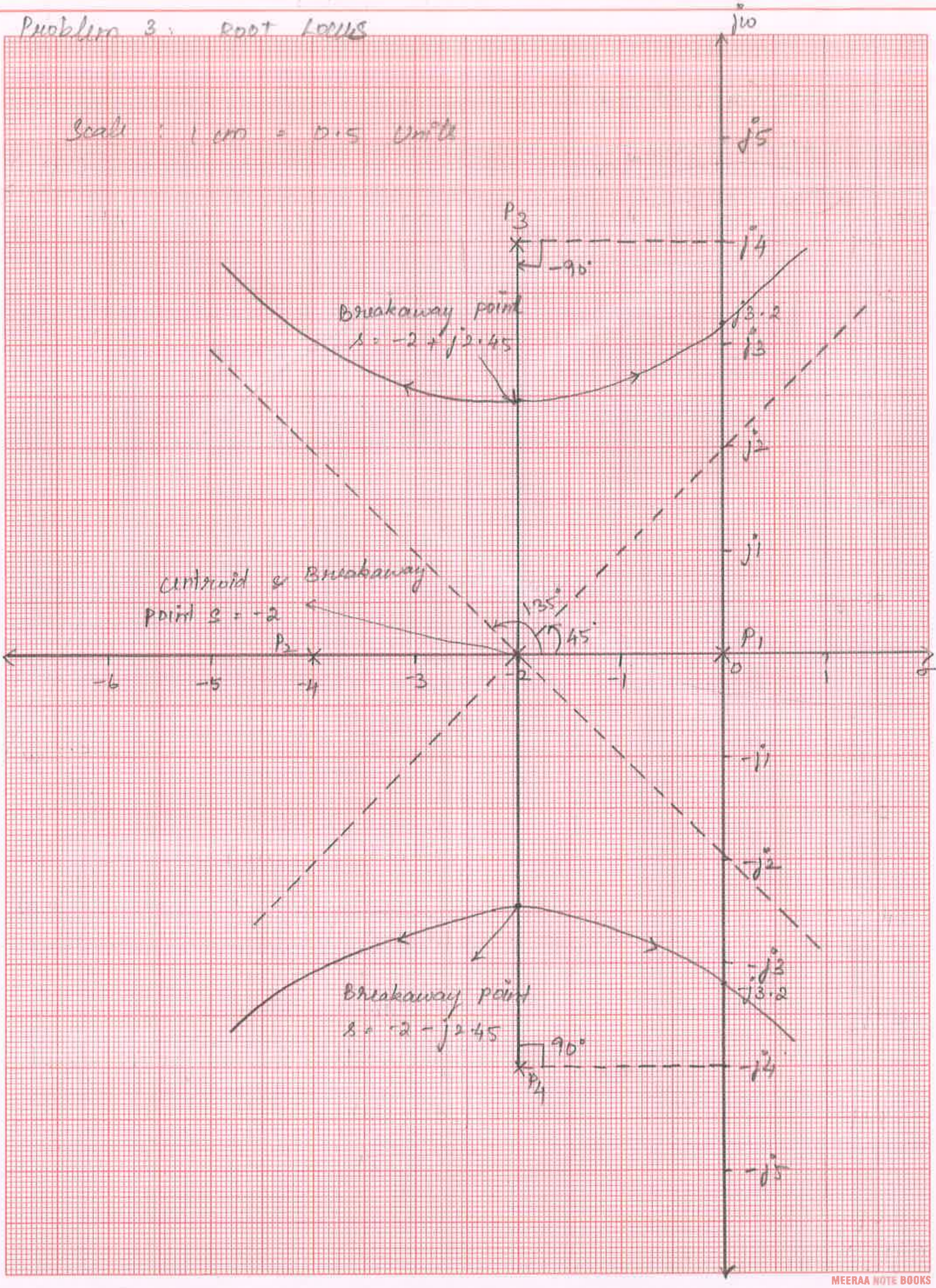
$= -90^\circ$

Angle of departure from complex pole P_4 $\Big\}$ $= -[P_3]$

$= +90^\circ$

Problem 3 : ROOT LOCUS

Scale : 1 cm = 0.5 Units



Step 6: To find crossing point on imaginary axis.

$C.E \Rightarrow s^4 + 8s^3 + 36s^2 + 80s + k = 0$

$s = j\omega$

$(j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80(j\omega) + k = 0$

$\omega^4 - 8\omega^3 j - 36\omega^2 + 80j\omega + k = 0$

Imaginary part

$-8\omega^3 j + 80j\omega = 0$

$8\omega^3 j = 80 j\omega$

$\omega^2 = 10$

$\omega = \pm\sqrt{10} = \pm 3.2$

$\boxed{\omega = \pm 3.2}$

Real part.

$\omega^4 - 36\omega^2 + k = 0$

$(\omega^2)^2 - 36\omega^2 + k = 0$

$k = 36(10) - (10)^2$

$= 360 - 100$

$\boxed{k = 260}$

RELATIVE STABILITY:

The relative stability indicates the closeness of the system to stable region. It is an indication of the strength of degree of stability.

GAIN MARGIN:

Gain margin is a factor by which the system gain can be increased to drive the system to the verge of instability.

$$\text{Gain margin}, k_g = \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}} = \frac{1}{G_A}$$

$$\text{Gain margin in db} = 20\log\frac{1}{|G(j\omega)H(j\omega)|} = 20\log\frac{1}{G_A}$$

$$= -20\log G_A$$

PHASE MARGIN :

Phase margin is defined as the amount of additional phase lag at gain cross over frequency required to bring the system to verge of instability.

$$\phi_{gc} = \angle G(j\omega) H(j\omega) \big|_{\omega = \omega_{gc}} \Rightarrow -180° + \gamma = \phi_{gc}$$

Phase margin, $\gamma = 180° + \phi_{gc}$.

Problem 1:

The open loop T.F of a system is $G(s) = \dfrac{K}{s(1+0.1s)(1+s)}$

i) Determine the value of K so that gain margin is 6db.

ii) Determine the value of K so that phase margin is 40°.

Sol :

i) To find K for specified gain margin

$$G(s) = \frac{K}{s(1+0.1s)(1+s)}$$

$s = j\omega$

$$G(j\omega) = \frac{K}{j\omega(1+j0.1\omega)(1+j\omega)}$$

$$= \frac{K}{j\omega(1+j1.1\omega - 0.1\omega^2)}$$

$$= \frac{K}{-1.1\omega^2 + j\omega(1-0.1\omega^2)}$$

Imaginary part $= 0$.

$\omega = \omega_{pc} \Rightarrow \omega_{pc}(1 - 0.1\omega^2) = 0$.

$$1 - 0.1\omega_{pc}^2 = 0$$

$$-0.1\omega_{pc}^2 = -1$$

$$\omega_{pc} = \frac{1}{\sqrt{0.1}} = 3.162 \text{ rad/sec}$$

$$|G(j\omega)|_{\omega=\omega_{pc}} = \left|\frac{K}{-1.1\omega^2}\right|_{\omega=\omega_{pc}}$$

$$= \frac{K}{1.1 \times 3.162^2} = 0.0909\,K$$

Gain margin = 6db.

$$20 \log K_g = 6$$

$$\log K_g = \frac{6}{20}$$

$$\text{Gain margin, } K_g = 10^{6/20} = 1.9953$$

By definition of GM,

$$GM,\ K_g = \frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}$$

$$1.9953 = \frac{1}{0.0909K}$$

$$K = \frac{1}{0.0909 \times 1.9953}$$

$$\boxed{K = 5.5135}$$

ii) To find K for specified phase margin.

$$G(s) = \frac{K}{s(1+0.1s)(1+s)}$$

$s = j\omega$

$$G(j\omega) = \frac{K}{j\omega(1+j0.1\omega)(1+j\omega)}$$

$$= \frac{K}{\omega\angle 90^\circ \sqrt{1+(0.1\omega)^2}\angle \tan^{-1}0.1\omega \sqrt{1+\omega^2}\angle \tan^{-1}\omega}$$

$$|G(j\omega)| = \frac{K}{\omega\sqrt{1+0.01\omega^2}\sqrt{1+\omega^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}0.1\omega - \tan^{-1}\omega \Rightarrow \phi_{gc}$$

P.M $\gamma = 180^\circ + \phi_{gc}$

$$40^\circ = 180^\circ - 90^\circ - \tan^{-1}0.1\omega_{gc} - \tan^{-1}\omega_{gc}.$$

$$\tan^{-1}0.1\omega_{gc} + \tan^{-1}\omega_{gc} = 180^\circ - 90^\circ - 40^\circ$$

$$\tan^{-1}0.1\omega_{gc} + \tan^{-1}\omega_{gc} = 50^\circ \qquad \left[\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right.$$

put tan on either. Side,

$$\tan\left[\tan^{-1}0.1\omega_{gc} + \tan^{-1}\omega_{gc}\right] = \tan 50^\circ$$

$$\frac{\tan(\tan^{-1}0.1\omega_{gc}) + \tan(\tan^{-1}\omega_{gc})}{1 - \tan(\tan^{-1}0.1\omega_{gc}) \times \tan(\tan^{-1}\omega_{gc})} = \tan 50$$

$$\frac{0.1\omega_{gc} + \omega_{gc}}{1 - 0.1\omega_{gc} \times \omega_{gc}} = 1.192 \Rightarrow \frac{1.1\omega_{gc}}{1 - 0.1\omega_{gc}^2} = 1.192$$

$$1.1\omega_{gc} = 1.192(1 - 0.1\omega_{gc}^2) \Rightarrow 0.1192\omega_{gc}^2 + 1.1\omega_{gc} - 1.192 = 0$$

$$\omega_{gc}^2 + \frac{1.1}{0.1192}\omega_{gc} - \frac{1.192}{0.1192} = 0$$

$$\omega_{gc}^2 + 9.228\omega_{gc} - 10 = 0$$

$$\omega_{gc} = \frac{-9.228 \pm \sqrt{9.228^2 + 4 \times 10}}{2} = \frac{-9.228 \pm 11.1873}{2}$$

Take $+^{ve}$ Value,

$$\omega_{gc} = \frac{-9.228 + 11.1873}{2} = 0.98 \text{ rad/sec}$$

At $\omega = \omega_{gc}$, $|G(j\omega)| = 1$

$$|G(j\omega)|_{\omega=\omega_{gc}} = \frac{K}{\omega_{gc}\sqrt{1+0.01\omega_{gc}^2}\sqrt{1+\omega_{gc}^2}} = 1$$

$$K = \omega_{gc}\sqrt{1+0.01\,\omega_{gc}^2}\sqrt{1+\omega_{gc}^2}$$

$$= 0.98\sqrt{1+0.01\times 0.98^2}\sqrt{1+0.98^2}$$

$$\boxed{K = 1.3787}$$

Result:

For a GM of 6 db , $K = 5.5135$

For a PM of $40°$, $K = 1.3787$.

NYQUIST STABILITY CRITERION:

Consider CLTF $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

C.E is $1+G(s)H(s) = 0$

Let $F(s) = 1+G(s)H(s)$

LTF $G(s)H(s)$ can be expressed as

$$G(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)} \quad \text{where } m \le n. \rightarrow ①$$

$$F(s) = 1+G(s)H(s) = 1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)}$$

$$= \frac{(s+P_1)(s+P_2)\cdots(s+P_n) + K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)}$$

$$= \frac{(s+z_1')(s+z_2')\cdots(s+z_n')}{(s+P_1)(s+P_2)\cdots(s+P_n)} \rightarrow ②$$

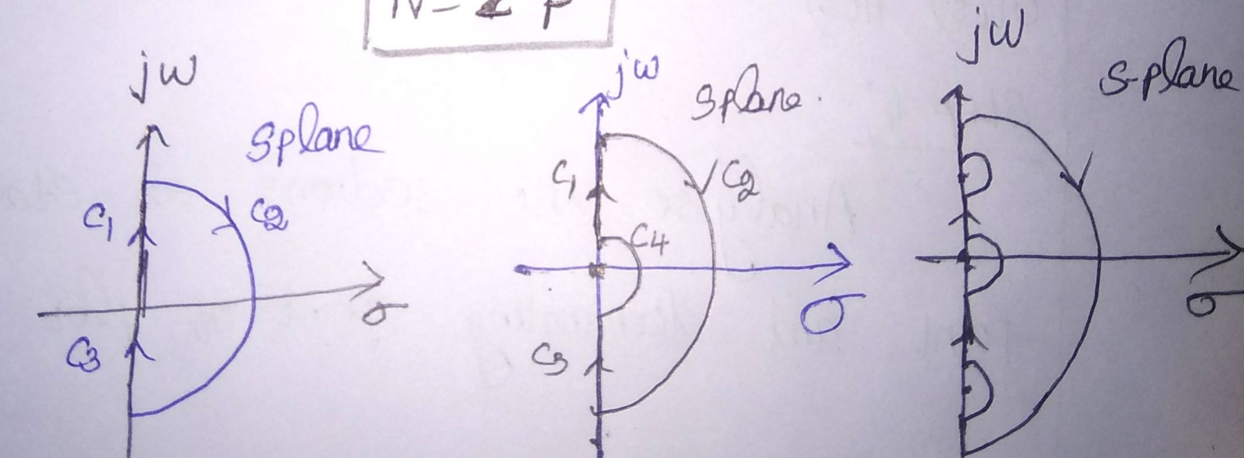
Nyquist plots

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from $-\infty$ to ∞.

Nyquist Stability Criterion

It states that if there are P poles and Z zeros are enclosed by the 's' plane closed path, then the corresponding $G(s)H(s)$ plane must encircle the origin $Z-P$ times. So we can write the number of encirclement N as

$$\boxed{N = Z - P}$$



Nyquist contour when there is no pole on imaginary axis.

When there are poles at origin

When there are poles on imaginary axis and at origin

Steps to solve Problems by Nyquist Criteria

Step 1 :-

Count how many number of poles of G(s) H(s) are in the right half of s plane (i.e) positive real part. This is the value of P.

Step 2 :

Decide the stability Criterion on as $N = -P$ how many times Nyquist plot should encircle $-1+j0$ point for absolute stability.

Step 3 :-

Select Nyquist plot as per function G(s) H(s)

Step 4 :

Analyse the sections as starting point and terminating point of plot.

Step 5:-

Mathematically find out ω_{pc} and intersection of Nyquist plot with Negative real axis by rationalizing $G(j\omega)\,H(j\omega)$

Step 6:-

With the knowledge of step 4 & 5, Sketch Nyquist plot.

step 7:-

Count the number of encirclements N of $(-1+j0)$ by Nyquist plot. If this matches with criterion decided in step 2, system is stable, otherwise unstable.

1) A unity feedback control system has $G(S) = \frac{10}{S(S+1)(S+2)}$. Draw Nyquist plot and

Comment on closed loop stability.

Solution

$$G(S)\,H(S) = \frac{10}{S(S+1)(S+2)}$$

$$H(S) = 1$$

Step 1:

No. of poles in right half $P=0$.

No pole of $G(S)\,H(S)$ is right half

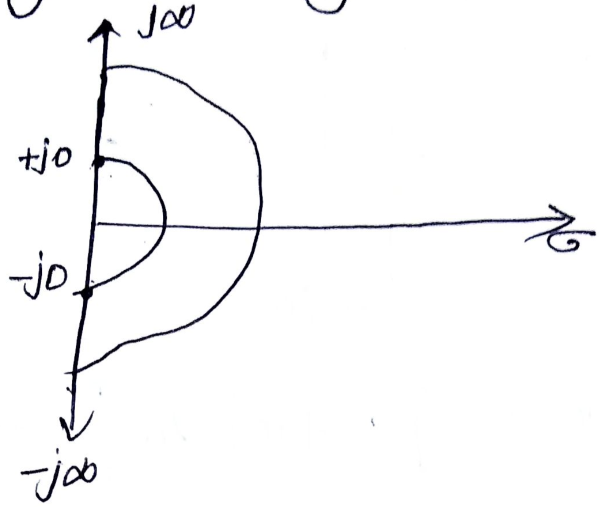
Step 2:

for stability $N = -P = 0$

Nyquist plot should not encircle $(-1+j0)$ point for absolute stability of this system.

Step 3:-

As there is one pole at origin, it should be bypassed by semicircle.



Step 4:-

$$G(j\omega)\,H(j\omega) = \frac{10}{j\omega(1+j\omega)(2+j\omega)}$$

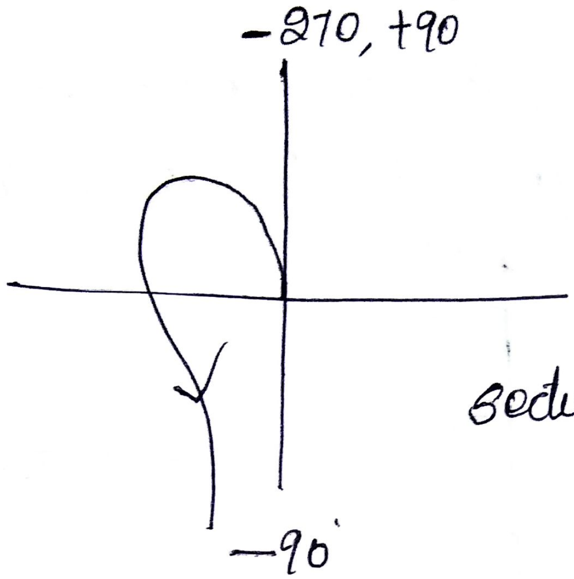
$$|G(j\omega)\,H(j\omega)| = \frac{10}{\omega\sqrt{1+\omega^2}\,\sqrt{4+\omega^2}}$$

$$\angle G(j\omega)\,H(j\omega) = -90 - \tan^{-1}\omega - \tan^{-1}\omega/2$$

Section : I $\boxed{s = +j\infty}$ to $\boxed{s = +j0}$ (ie) $\omega \to \infty$ to $\omega = 0$.

Starting point $\rightarrow \omega \to \infty \Rightarrow 0\angle -270^\circ$

Terminating point $\rightarrow \omega \to 0 \Rightarrow \infty\angle -90^\circ$

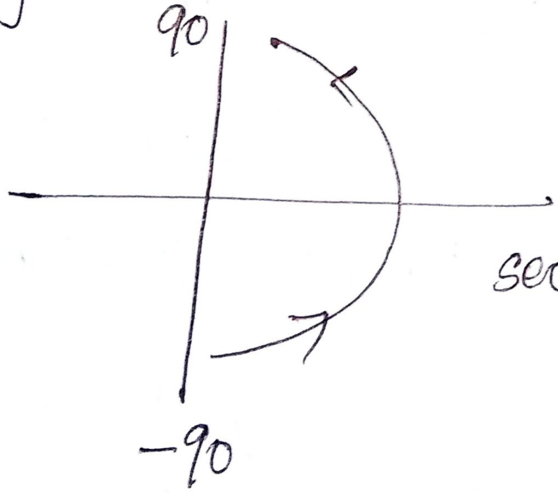


Section - I mapping

Section : II $\boxed{s = +j0}$ to $\boxed{s = -j0}$ (i.e) $\omega \to 0$ to $\omega \to -0$

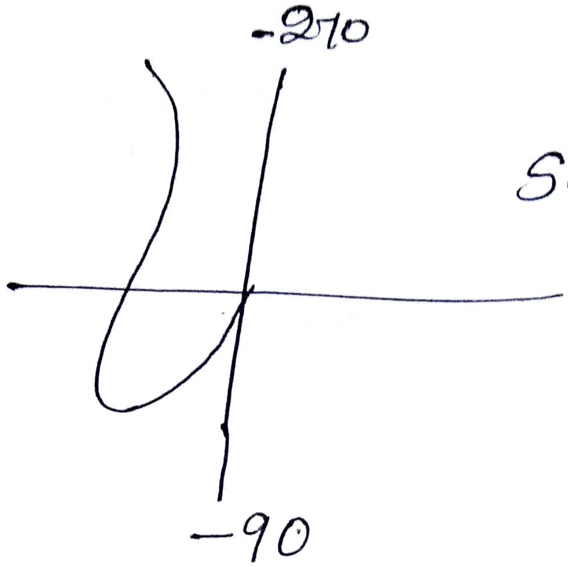
Starting point :- $\omega \to +0 \Rightarrow \infty \angle -90°$

Terminating point :- $\omega \to -0 \Rightarrow \infty \angle +90°$



Section II mapping

Section III : It is the Mirror image of section I about real axis



Section III mapping

Section IV :- It is the origin and not required to analysed.

Step 5: Find out intersection with negative real axis, Rationalise $G(j\omega)\,H(j\omega)$.

$$G(j\omega)\,H(j\omega) = \frac{10(-j\omega)(1-j\omega)(2-j\omega)}{(j\omega)(-j\omega)(1+j\omega)(1-j\omega)(2+j\omega)(2-j\omega)}$$

$$= \frac{-30\omega^2 - 10j\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

Separate real & Imag part.

$$G(j\omega)\,H(j\omega) = \frac{-30\omega^2}{\omega^2(1+\omega^2)(4+\omega^2)} - \frac{10j\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

At $w = w_{pc}$ $\Rightarrow$ imaginary part is zero.

$$10w(2-w^2) = 0$$

$10w = 0$ $\qquad$ $2-w^2=0$

$w = 0$ $\qquad$ $w^2 = 2$

$w = \sqrt{2}$

Substitute $w_{pc} = \sqrt{2}$ in real part

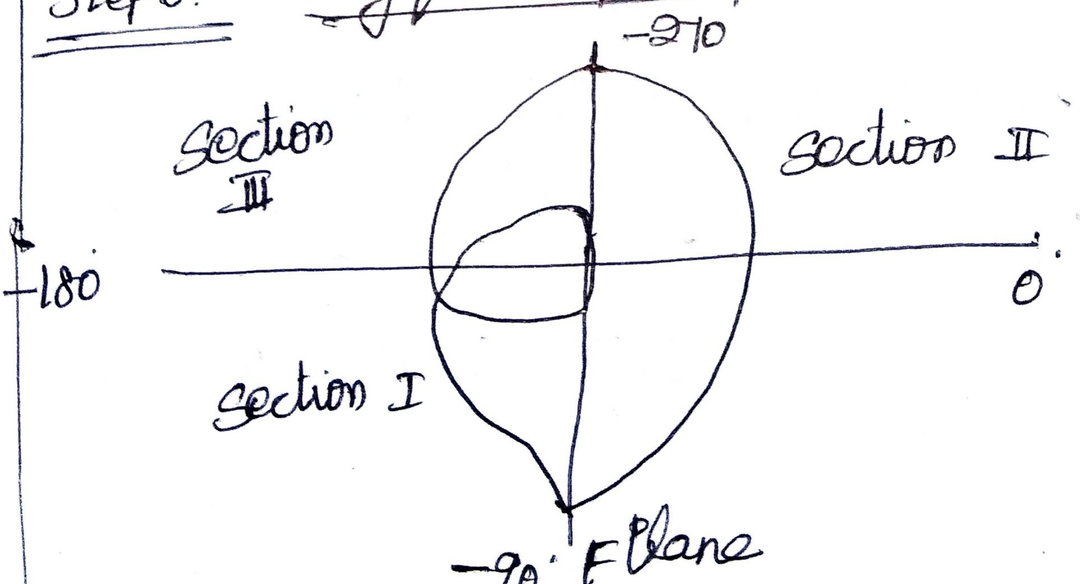
$$G(jw)\,H(jw)\Big|_{w=w_{pc}} = \frac{-30\times 2}{2(1+2)(4+2)} + j0$$

$$= -1.66 + j0.$$

$$\text{Gain Margin} = \frac{1}{|-1.66+j0|} = \frac{1}{1.66} = 0.6 = -4.43\,dB$$

As GM is negative, system is unstable because critical point is enclosed.

Step 6: Nyquist plot



$N = +2$

Step 7:-

The Number of encirclement of $-1+j0$ are $\boxed{N=+2}$. as per step 2:- $\boxed{N=0}$. Hence it does not match. given system is unstable.

According to Mapping theorem. $N=Z-P$

$2=Z-0$

$\boxed{Z=2}$

Actually there are 2 zeros of $1+G(s)H(s)$ encircled by Nyquist path.

UNIT 5

Control System Analysis using State Variable

Methods

Basically, there are 2 approaches, to the analysis and design of control systems.

1) Transfer function approach
2) State Variable Approach.

Transfer function Approach	State Variable Approach
1) Transfer function approach is also called conventional approach	1. State variable Approach is called Modern Approach.
2) It is applicable to linear time Invariant systems. It is generally limited to single input single output systems	2) It is applicable to linear as well as non linear, Time invariant as well as Time variant, single i/p single o/p as well as multi i/p multi o/p systems

3) In this initial Conditions are neglected	3) They are considered
4) It is basically a frequency domain approach	4) It is a time domain Approach.
5) It is based on Trial & error procedure	5) It is not based on Trial and error procedure
6) Only i/p, o/p & Error Signals are Considered important. The i/p & o/p Variables must be Measurable	6) State variable need not represent physical variables They need not be measurable & observable
7) Internal variables cannot be feedback	7) State variables can be feedback.
8) Transfer function of a System is unique.	8) State model of a System is not unique.

State Variable Representation

Set of variables which describe the System at any time instant are called state variables In the state variable formulation of a system, in general, a system consists of m inputs, P outputs n state variables

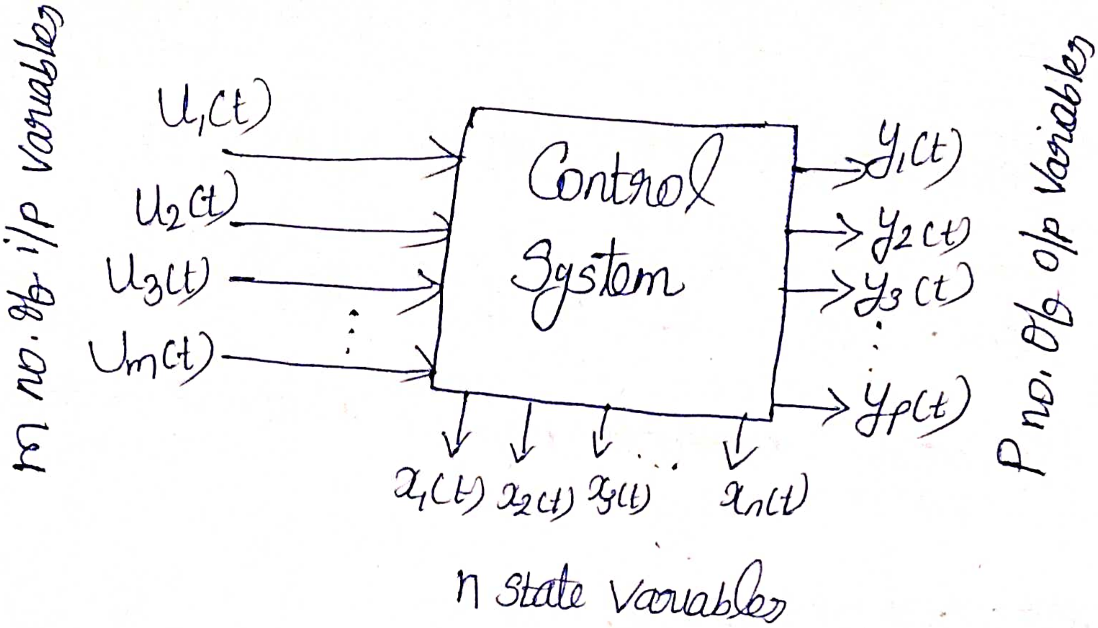


Fig: State space representation of a System

Different variables may be represented by the vectors

Input vector $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$ Output vector $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

State variable vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

State Equations

State Variable Representation can be arranged in the form of n no. of 1st order differential equations as shown below.

$$\frac{dx_1}{dt} = x_1' = f_1(x_1, x_2, \dots x_n ; u_1, u_2, \dots u_m)$$

$$\frac{dx_2}{dt} = x_2' = f_2(x_1, x_2, \ldots x_n ; u_1, u_2 \ldots u_m)$$

$$\frac{dx_n}{dt} = x_n' = f_n(x_1 x_2, \ldots x_n ; u_1, u_2 \ldots u_m)$$

'n' no. of differential equations may be written in vector notation as

$$x'(t) = f[x(t), u(t)]$$

* The set of all possible values which the i/p vector u(t) can have at a time t form the i/p space of the system.

* The set of all possible values which the o/p vector y(t) can have at a time t forms the the o/p space of the system.

* The set of all possible values which the state vector x(t) can have at a time t form the state space of system.

State Model of Linear System

The state model of a system consists of state eqn & o/p equation.

The state equation of a system is a function of state variables & i/p's.

For linear time invariant systems, the first derivative of state variables can be expressed as linear combination of state variables and inputs.

$$x_1' = (a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n) + (b_{11}u_1 + b_{12}u_2 + \dots b_{1m}u_m)$$

$$x_2' = (a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n) + (b_{21}u_1 + b_{22}u_2 + \dots b_{2m}u_m)$$

$$\vdots$$

$$x_n' = (a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n) + (b_{n1}u_1 + b_{n2}u_2 + \dots b_{nm}u_m)$$

Where coefficients a_{ij} & b_{ij} are constants

In the matrix form, above equations can be written as

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$x'(t) = Ax(t) + Bu(t) \quad \text{—} \; (1)$$

Where

$x(t)$ is state vector of order $n \times 1$

$u(t)$ is input vector of order $m \times 1$

A = System matrix with order $n \times n$

B = Input matrix with order $n \times m$.

Eqn -(1) is called state equation of LIT system.

The o/p at any time are functions of state variables and input

Output vector $y(t) = f[x(t), u(t)]$

The o/p variables can be expressed as linear combination of state variables and inputs.

$$y_1' = (C_{11}x_1 + C_{12}x_2 + \dots C_{1n}x_n) + (d_{11}u_1 + d_{12}u_2 + \dots d_{1m}u_m)$$

$$y_2' = (C_{21}x_1 + C_{22}x_2 + \dots C_{2n}x_n) + (d_{21}u_1 + d_{22}u_2 + \dots d_{2m}u_m)$$

$$\vdots$$

$$y_n' = (C_{p1}x_1 + C_{p2}x_2 + \dots C_{pn}x_n) + (d_{p1}u_1 + d_{p2}u_2 + \dots d_{pm}u_m)$$

In matrix form, above eqns can be written as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & & & \\ C_{p1} & C_{p2} & \dots & C_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \dots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$Y(t) = C\,X(t) + D\,u(t) \longrightarrow (2)$$

Where $X(t)$ - State vector of order $n \times 1$

$U(t)$ - Input vector of order $m \times 2$

$Y(t)$ = Output vector of order $P \times 1$

C = Output matrix of order $P \times n$

D = Transmission matrix of order $P \times m$.

Eqn - ② is called Output equation of LTI system. The state equation O/P eqn together called as State model of the system. Hence the state model of the LTI system is given by

$$x'(t) = Ax(t) + Bu(t) \quad \text{- State equation}$$

$$y(t) = Cx(t) + Du(t) \quad \text{- O/P equation}$$

Conversion of state variables models to transfer functions

① Obtain T.F if $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(s) = C[sI - A]^{-1} B$$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1}$$

$$= \frac{1}{(s+5)(s+1)+3} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$= \frac{1}{s^2+6s+8} \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+1}{s^2+6s+8} & \frac{-1}{s^2+6s+8} \\ \frac{3}{s^2+6s+8} & \frac{s+5}{s^2+6s+8} \end{bmatrix}$$

$$C[sI-A]^{-1}B = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{s+1}{(s+2)(s+4)} & \frac{-1}{(s+2)(s+4)} \\ \frac{3}{(s+2)(s+4)} & \frac{s+5}{(s+2)(s+4)} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+1+6}{(s+2)(s+4)} & \frac{-1+2s+10}{(s+2)(s+4)} \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \frac{1}{(s+2)(s+4)} \begin{bmatrix} s+7 & 2s+9 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \frac{1}{(s+2)(s+4)} \begin{bmatrix} 2s+14+10s+45 \end{bmatrix}$$

$$= \frac{12s+59}{(s+2)(s+4)}$$

Conversion of transfer functions to state variable Models.

① Obtain state Model for the system whose transfer function is given as $\cancel{\frac{Y(S)}{U(S)} = \frac{5S+6}{S^2+2S+3}}$

$$\frac{Y(S)}{U(S)} = \frac{10}{S^3+4S^2+2S+1}$$

Solution

$$Y(S)\left[S^3+4S^2+2S+1\right] = 10U(S)$$

$$S^3 = \frac{d^3}{dt^3} \qquad S^2 = \frac{d^2}{dt^2} \qquad S = \frac{d}{dt}$$

$$\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y - 10U(S) = 0$$

Let

$$x_1 = y, \quad x_2 = \frac{dy}{dt} = x_1', \quad x_3 = \frac{d^2y}{dt^2} = x_2', \quad x_4 = \frac{d^3y}{dt^3} = x_3'$$

$$x_3' + 4x_3 + 2x_2 + x_1 - 10u = 0$$

$$\Rightarrow x_3' = -4x_3 + 2x_2 + x_1 - 10u$$

$$x_2' = x_3$$

$$x_1' = x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} [u]$$

Output equation

$$y = x_1$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution of State equations

A Linear time Invariant system is characterised by homogeneous state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Compute the solution of homogeneous equation assuming initial vector $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

$$e^{At} = \phi(t) = L^{-1}[SI-A]^{-1}$$

$$SI-A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix}$$

$$[SI-A]^{-1} = \frac{1}{(S-1)^2} \begin{bmatrix} S-1 & 0 \\ 1 & S-1 \end{bmatrix} = \begin{bmatrix} 1/S-1 & 0 \\ 1/(S-1)^2 & 1/S-1 \end{bmatrix}$$

$$e^{At} = \phi(t) = L^{-1}[SI-A]^{-1} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

The Solution of the State equation is

$$X(t) = e^{At} X_0 = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

Concept of Controllability

Controllability of Linear Systems

A system is said to be completely state Controllable at time t_0, if it is possible by means of an unconstrained control vector $u(t)$ to transfo the system from an initial state $x(t_0)$ to any other desired state in a finite interval of time.

① Evaluate Controllability of System $\dot{x} = AX + BU$,

$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $x(t)$ is n dimensional state vector.

Solution

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad n = 2$$

Controllability matrix $S = \begin{bmatrix} B & AB \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$|S| = 0$, Rank $= 1$

$|S| = 0$ & Rank $\neq n$, the given System is not Controllable.

(2) $\dot{X} = AX + BU$, $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

$n = 2$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Controllable matrix $S = \begin{bmatrix} B & AB \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$|S| = -1$ rank $= 2$

System is controllable

Observability

A System is said to be completely observable if every state $x(t_0)$ can be completely identified by measurements of outputs $Y(t)$ over a finite time interval.

If the system is not completely observable means that few of its state variables are not practically measurable.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$Y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q_0 = \begin{bmatrix} C^T & A^T C^T \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad |Q_0| = 1 \neq 0$$

Rank $= 2 = n$

$\therefore$ The system is observable.

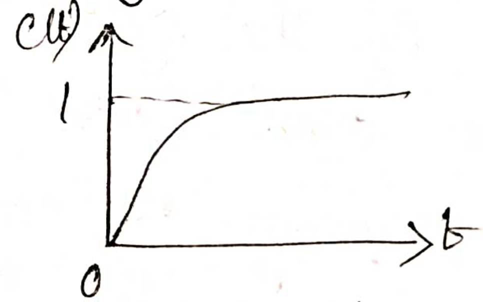
Equivalence between transfer function and state variable Representation

To convert a transfer function into state equations in phase variable form, we first convert the transfer function to a differential equation by cross multiplying and taking the inverse Laplace transform, assuming zero initial conditions.

Stability of linear Systems.

A System is said to be stable, if its output is under control. otherwise, it is said to be unstable. A stable System produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.



Types of Systems based on stability

* Absolutely stable system
* Conditionally stable system.
* Marginally stable system.

Absolutely stable system.

If the system is stable for all the range of system Component Values, then it is known as the absolutely stable system.

Conditionally stable system.

If the system is stable for a certain range of system component values, then it is known as conditionally stable system.

Marginally stable system.

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as Marginally stable system.

State space Representation of Digital (or) Discrete systems.

The state space representation of discrete system is exactly similar to the state space representation of Continuous time systems.

In Continuous systems, the state equations are derived generally from the differential equations of the system. While in discrete time systems the state equations are obtained from the difference equations

$$x[k+1] = f\{x(k), u(k), k\}$$

1) Obtain the state space of the difference equation

$$6y(k+3) + y(k+2) + 5y(k+1) + 6y(k) = u(k)$$

Solution:-

The order of the equation is $n=3$. Select lowest order term of $y(k)$ as first variable

$$x_1(k) = y(k)$$

$$x_1(k+1) = y(k+1) = x_2(k) \longrightarrow (1)$$

$$x_2(k+1) = y(k+2) = x_3(k) \longrightarrow (2)$$

$$x_3(k+1) = y(k+3)$$

$$\therefore 6x_3(k+1) + x_3(k) + 5x_2(k) + 6x_1(k) = u(k)$$

$$x_3(k+1) = -x_1(k) - \frac{5}{6}x_2(k) - \frac{1}{6}x_3(k) + \frac{1}{6}u(k) \longrightarrow (3)$$

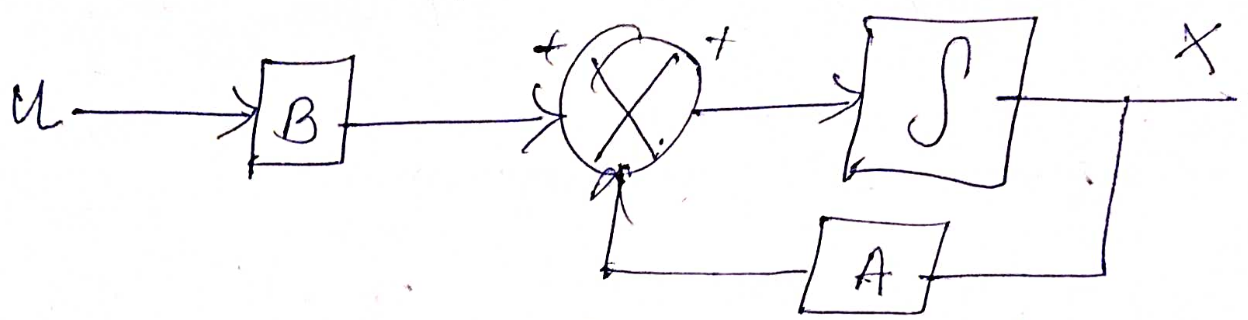
The output equation is

$$y(k) = x_1(k)$$

Digital Control System using state feedback

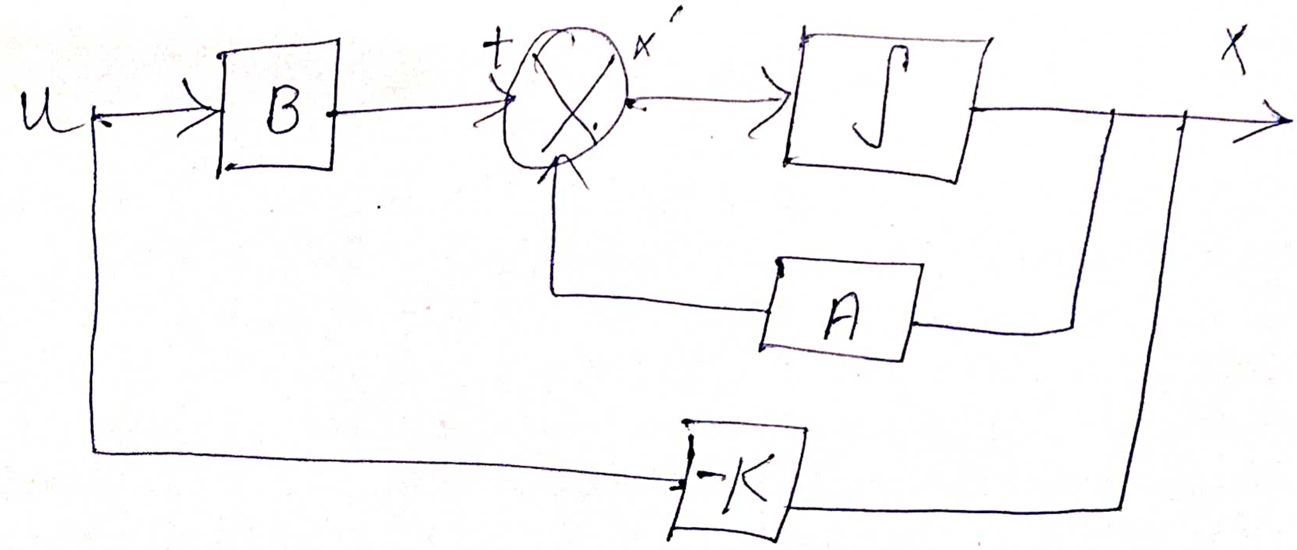
Consider a control system described by following state equation

$$x' = Ax + BU$$



open loop control System.

In closed loop control system, system state x is fed back to control signal u. Such a system is called as state feedback.



$$x' = Ax + B(-Kx) = (A-BK)x$$

Hence we can write the State Model

as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \frac{-5}{6} & \frac{-1}{6} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \end{bmatrix} u(k)$$

$$X(k+1) = G\,X(k) + H\,u(k)$$

$$Y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \\ X_3(k) \end{bmatrix}$$

$$Y(k) = C\,X(k) + D\,u(k)$$